Assignment 4 (Deadline: 3 April, 3.30 pm)

Problem 1. Solve the following two questions on trees (Source: *Discrete Mahematics: elementary and beyond*):

- Let G be a tree, which we consider as the network of roads in a medieval country, with castles as nodes. The King lives at node r. On a certain day, the lord of each castle sets out to visit the King. Argue carefully that soon after they leave their castles, there will be exactly one lord on each edge.
- If we delete a node v from a tree (together with all edges that end there), we get a graph whose connected components are trees. We call these connected components the branches at node v. Prove that every tree has a node such that every branch at this node contains at most half the nodes of the tree.

Problem 2. Prove that in a connected graph G with at least three vertices, any two longest paths have a vertex in common.

Problem 3. (Graphic matroid) Let (V, I) and (V, J) be two forests on the same vertex set V where $|I| < |J|$. Show that there is an edge $j \in J$ such that $(V, I \cup \{j\})$ is also a forest.

Problem 4. Let K_n be the complete graph on the vertices $[n]$ and $\mathbb{F}$ be a field. Let $x_1, x_2, \ldots, x_n$ be the standard basis for $\mathbb{F}^n$. For each $1 \leq i < j \leq n$, we associate a vector in $\mathbb{F}^n$ to the edge $e = \{i, j\}$ of K_n given by $x_e = x_i - x_j$. Show that the set of vectors $x_{e_1}, x_{e_2}, \ldots, x_{e_k}$ are linearly independent if and only if the edges $e_1, e_2, \ldots, e_k$ do not contain a cycle in K_n.

Problem 5. We know that for any n, the set of *all* transpositions (2-cycles) generates the symmetric group $\mathfrak{S}_n$. We associate a graph on the vertex set $[n]$ to a set of transpositions by identifying the transposition $(i\ j)$ with the edge joining the vertices i and j. Show that a set of transpositions generates $\mathfrak{S}_n$ if and only if the corresponding graph on $[n]$ is connected.

Problem 6. Let G be a group and $S \subseteq G$ be a subset of G. Construct a simple, undirected graph Γ with vertex set G and an edge between all pairs of the form $\{g, sg\}$ and $\{g, s^{-1}g\}$ for $g \in G, s \in S$. Show that S generates G if and only if Γ is a connected graph. In general, the connected components of Γ give us a partition of G. Can you describe this partition?

Problem 7. Let $\mathcal{S}$ be a collection of sets. Construct a graph with $\mathcal{S}$ as vertices by setting two sets in $\mathcal{S}$ to be adjacent if they intersect. Show that any simple graph G can be seen as such a graph.

Problem 8. Here is an alternative definition of matroid via circuits. Prove that this definition is *equivalent* with the definition of matroid via independent sets (show how to get circuits from independent sets and vice versa).

A matroid M is a pair $(E, \mathcal{C})$ consisting of a finite set E and a collection $\mathcal{C}$ of subsets of E satisfying:

- $\emptyset \notin \mathcal{C}$.
- If $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$
- If C_1, C_2 are distinct members of $\mathcal{C}$ and $e \in C_1 \cap C_2$, then there is a member C_3 of $\mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) - e$

Problem 9. Let T and T' be two distinct trees on the same vertex set. Let e be an edge that is in T but not T'. Show that there exists an edge e' that is in T' but not T such that $T' + e - e'$ (adding e to T' and removing e') is also a tree. Use this to prove that for a weighted connected graph with distinct edge weights, there is a unique minimal spanning tree.

Problem 10. For an undirected graph on the vertices $[n]$, if the adjacency matrix is $A_{n \times n}$, then the $(i,j)^{th}$ entry of A^m is the number of walks of length m from vertex i to j. Define the adjacency matrix A for a directed graph in such a way that the $(i,j)^{th}$ entry of A^m is the number of directed walks of length m from vertex i to j.

Problem 11. Show that the number of spanning trees of the complete bipartite graph $K_{2,n}$ is $n \cdot 2^{n-1}$.

Problem 12. Let G be a bipartite graph with partitions X and Y, and suppose that the degree of each vertex in X is greater than the degree of any vertex in Y. Prove that the graph has a matching covering every vertex in X.