Discrete Mathematics Assignment 3

Due Date: March 16, 2023

Assignment on Möbius Inversion.

1. Suppose $P_1 = (S_1, \prec_1)$ and $P_2 = (S_2, \prec_2)$ are finite posets. Define their product poset as $P(S_1 \times S_2, \prec)$ where $(a_1, a_2) \prec (b_1, b_2)$ iff $a_1 \prec_1 b_1$ and $a_2 \prec_2 b_2$. Show that the Möbius function μ of P is $\mu((a, b), (c, d)) = \mu_1(a, c) \cdot \mu_2(b, d)$, where μ_i is the Möbius functions of $P_i, i = 1, 2$.

2. Show that the subset poset $(2^{[n]}, \subseteq)$ is isomorphic to the boolean strings poset $(B^n, \prec)$ where $B = \{0, 1\}$ is the boolean poset and B^n is its n-fold product. Hence, derive the Möbius function of the subset poset to be $\mu(I, J) = (-1)^{|J \setminus I|}$ for $I \subseteq J$.

3. Let $f, g : 2^{[n]} \to \mathbb{R}$ be real-valued functions on subsets of $[n]$ such that $g(J) = \sum_{I \supseteq J} f(I)$ for every $J \subseteq [n]$. Prove using the Möbius inversion formula for the subset poset that $f(J) = \sum_{I \supseteq J} (-1)^{|I \setminus J|} \cdot g(I)$.

4. For the divisibility poset $([n], \leq)$, where $a \leq b$ iff a divides b, find the Möbius function $\mu(a, b)$ for $a, b \in [n]$. Using that show that if $g(m) = \sum_{n|m} f(n)$ then $f(m) = \sum_{n|m} \mu_c(m/n) \cdot g(n)$, where $\mu_c(t)$ is the classical Möbius function defined as $\mu_c(t) = (-1)^k$ if t is a product of k distinct primes, for $k \geq 0$, and is defined to be zero otherwise.

5. Let $P = (S, \prec)$ be an n-element poset and $x_1, x_2, \ldots, x_n$ be a total ordering of S that is a *linear extension* of P. Let $I(P)$ denote the incidence algebra of P and recall the homomorphism φ defined in class from $I(P)$ to $n \times n$ matrices over the reals:

$$\varphi : f \mapsto M_f$$

where the $(i, j)^{th}$ entry of M_f is $f(x_i, x_j)$ for $1 \leq i, j \leq n$. The matrix M_f is upper triangular.

- Write $M_f = D - N$ where D is a diagonal matrix and N is a *strictly* upper triangular matrix. That means, N is upper triangular and $N(i, i) = 0$ for all i. Show that $N^n = 0$.

- Show that $\varphi^{-1}(D)$ and $\varphi^{-1}(N)$ are defined in $I(P)$.
- Show that $\varphi^{-1}(D)$ has an inverse in $I(P)$ if and only if D is invertible.
- Suppose D is invertible. Show that $\varphi^{-1}(D^{-1}N)$ is defined, and $D^{-1}N = M$ is strictly upper triangular.
- Show that $(I - M)^{-1}$ is $I + M + M^2 + \cdots M^{n-1}$. Hence prove that if D is invertible there is a $g \in I(P)$ such that $f * g = \delta$, where $\delta \in I(P)$ is the identity element.