Diff Eqns Jan–Apr 2024 Mid-term Examination 2024-02-28 09:30–12:30

Answer all the questions. There are problems worth 160 marks in this exam. Contribution towards final grade: (your marks)/5 or 30 whichever is lower.

1. (10 points) Find (with justification) two different solutions to the system $\dot{x} = 2\sqrt{|x|}, x(0) = 0$.

2. (10 points) Let $v(x)$ be a continuous vector field on $\mathbb{R}$. Can two integral curves of v intersect? (Justify.)

3. (25 points) Sketch the phase curves of the two-dimensional system $\dot{\mathbf{x}} = A\mathbf{x}$ where A is a 2×2 matrix with determinant $\delta > 0$ and trace $t = 0$. (Your sketch can be out-of-scale etc, but, there should be accompanying text that clearly explains what the phase curves look like. You must mark the increasing direction of t on the phase curves.)

4. Let A be a 2×2 Jordan block matrix with eigenvalue λ. Solve the IVP $\dot{\mathbf{x}} = A\mathbf{x}$ with $\mathbf{x}(0) = [1 \;\; 1]^T$ in the following two cases.
   (a) (15 points) $\lambda < 0$.
   (b) (10 points) $\lambda = 0$.

5. (20 points) Determine (with proof) whether there exists a one-parameter group $\{g^t\}$ of diffeomorphisms $\mathbb{R} \longrightarrow \mathbb{R}$ that correspond to the autonomous system $\dot{x} = x^2$.

6. (10 points) Show that the solution to the IVP $\dot{x} = (1-x)x - \frac{1}{4}, x(0) = 1$ tends to $\frac{1}{2}$ as $t \longrightarrow +\infty$.

7. Let $V \subseteq \mathbb{R}^n$ be a non-empty open connected subset and $\{g^t\}$ a one-parameter group of diffeomorphisms on V. Let $\phi : V \longrightarrow W \subseteq \mathbb{R}^n$ be a diffeomorphism. Let $\{h^t := \phi g^t \phi^{-1}\}$ be a induced one-parameter group of diffeomorphisms on W. Assume that h^t is the map $(y_1, \dots, y_n) \mapsto (y_1 + t, y_2, \dots, y_n)$
   (a) (5 points) Describe the phase curves of $\{h^t\}$.
   (b) (15 points) Let $x^0 \in V$ and $x^k, k > 0$ be a sequence in V with limit x^0. Show that for each $t \in \mathbb{R}$, $g^t x^k$ converges to $g^t x^0$.

8. (10 points) Describe qualitatively (in words and with a rough sketch) the graph of the solution to the system $\dot{x} = \sin x, x(0) = 1$.

9. (10 points) Consider the system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

   Show that there is a proper subspace V in $\mathbb{R}^2$ such that if $\phi(t)$ is a solution to the above system with initial condition $\phi(0) \in V$, $\phi(t) \in V$ for all t.

10. Consider the system

$$\dot{x}_1 = x_2 + x_1(1 - x_1^2 - x_2^2)$$
$$\dot{x}_2 = -x_1 + x_2(1 - x_1^2 - x_2^2)$$

   (a) (10 points) Show that this system transforms to the following in polar coordinates: $\dot{r} = r(1-r^2)$, $\dot{\theta} = -1$.
   (b) (10 points) Describe the phase curves (words and sketch, identifying equilibrium points, increasing direction of t, etc.) in polar and cartesian coordinates.