Diff Eqns Jan–Apr 2024 Final Examination 2024-05-06 09:30–12:30

Answer all the questions. There are problems worth 120 marks in this exam. Contribution towards final grade: (your marks)/3

1. Let a, b, k, l be positive real numbers. Consider the the predator-pray (or Lotka-Volterra) system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = v(x, y) := \begin{bmatrix} kx - axy \\ -ly + bxy \end{bmatrix}$$

on the phase space $U := \{(x, y) \mid x > 0; y > 0\}$.

   (a) (5 points) Write (x^*, y^*) for the equilibrium point. Determine x^* and y^*.

   (b) (5 points) Show that at each (x, y), the slope of the tangent to the phase curve through (x, y) is

$$\frac{-by(x - x^*)}{ax(y - y^*)},$$

   (where we interpret the expression as a limit when the denominator is zero).

   (c) (20 points) Find functions $f(x)$ and $g(y)$ such that, with

$$F(x, y) := \int_{x^*}^{x} f(\xi) d\xi + \int_{y^*}^{y} g(\eta) d\eta$$

   the directional derivative $D_v F$ of F in the direction of v is 0 everywhere. (Hint : separation of variables)

   (d) (5 points) Using the above step, show that the given system is Lyapunov-stable and not asymptotically stable.

   (e) (10 points) Let $(x_0, y_0) \in U$. Show that $F(x, y) = F(x_0, y_0)$ is the equation of the phase curve through (x_0, y_0).

2. Consider the system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -y - xy \\ x + x^2 \end{bmatrix}$$

   (a) (10 points) Determine whether $F(x, y) = x^2 + y^2$ is a Lyapunov function.

   (b) (5 points) Determine the nature of stability at the origin: unstable / Lyapunov-stable / asymptotically stable.

3. Consider the system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -x \\ y + x^2 \end{bmatrix}$$

on $\mathbb{R}^2$.

   (a) (5 points) Determine the stable and the unstable subspaces of the linearization.

   (b) (10 points) Using Taylor series expansion, find the equation of the stable manifold.

   (c) (5 points) Show that the y-axis is the unstable manifold, i.e., the set $\{p \mid \lim_{t \to -\infty} g^t p = 0\}$.

4. (10 points) Find the maximal interval of existence of the solution to

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \frac{1}{2x} \\ y^2 \end{bmatrix}, \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

5. (15 points) Let $\dot{\mathbf{x}} = v(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$ be a system. Let $\mathbf{x}_0 \in \mathbb{R}^n$ and I be the maximal interval for the solution $\psi(t)$ to the above system with initial condition $\mathbf{x}(0) = \mathbf{x}^0$.

   Prove or disprove the following statement. If $\{\psi(t) \mid t \in I\}$ is closed in $\mathbb{R}^n$ but not a point, then ψ is a periodic solution.

6. (15 points) Let $\dot{\mathbf{x}} = A\mathbf{x}$ and $\dot{\mathbf{x}} = B\mathbf{x}$ be two linear systems on $\mathbb{R}^n$ and $\{f^t\}, \{g^t\}$ be the corresponding flows. Suppose that $h : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a diffeomorphism such that $h(0) = 0$ and such that for all $\mathbf{x} \in \mathbb{R}^n$ and for all t, $g^t h(\mathbf{x}) = h(f^t\mathbf{x})$. Show that there is a linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $g^t T(\mathbf{x}) = T(f^t\mathbf{x})$.