Calculus

Midsemester exam

07/10/23, 9:30-12:30

You may use Prof. Ramdas' notes or your class notes . No other sources are permitted. In a multi-part question, the result of one part may be used in a succeeding part even if you have not been able to derive it.

1. ($2.5 \times 4 = 10$ *points*) True or false? Give either a proof or a counterexample to justify your answer.

   (a) An increasing function $f : [a, b] \to \mathbb{R}$ is Riemann integrable.

   (b) If $E \subset \mathbb{R}$ has measure 0 then either E is bounded or countable.

   (c) Let $A \subset \mathbb{R}^n$ be a closed rectangle, and $C \subset A$ a subset of measure 0. Then $\int_A \chi_C = 0$, where χ_C is the indicator function of C.

   (d) Let $Q \subset \mathbb{R}^2$ be the rectangle $[0, 1] \times [0, 1]$. Define $f : [0, 1] \times [0, 1] \to \mathbb{R}$ as $f(x, y) = 0$ if x or y is irrational, and $f(x, y) = 1/n$ if $x, y \in \mathbb{Q}$ and $x = m/n$ with $(m, n) = 1$. Then, since $\int_Q f$ exists and is 0, each of the iterated integrals exists and is 0.

2. (*5+5=10 points*) Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be a continuous, and suppose $D_2 f$ exists and is continuous.

   (a) Define $F(y) := \int_a^b f(x, y)dx$. Show that $F'(y) = \int_a^b D_2 f(x, y)dx$.

   (b) Let $G(x, y) := \int_a^x f(t, y)dt$, and suppose that $g : [c, d] \to [a, b]$ is differentiable. Compute $D_1 G$ and $D_2 G$. Set $H(x) = \int_a^{g(x)} f(t, x)dt = G(g(x), x)$. Find $H'(x)$.

3. (*10 points*) Denote by p_k the k-th prime number, and let

$$S_k = \{(m/p_k, n/p_k) | m, n \in \mathbb{Z}, 0 < m, n < p_k\}$$

   (a) Show that $S := \cup_{k=1}^{\infty} S_k$ is dense in $Q = [0, 1] \times [0, 1]$, but any line parallel to the coordinate axes contains only finitely many points of S.

   (b) Define $f : Q \to \mathbb{R}$ as $f(x, y) = 0$ if $(x, y) \in S$, and $f(x, y) = 1$ is $(x, y) \in Q - S$. Calculate the two iterated integrals of f on Q as well as the integral $\int_Q f$ (or argue that they do not exist).

4. (*10 points*) Let S be a subset of a rectangle $R \subset \mathbb{R}^n$. Let P be a partition of R consisting of sub-rectangles R_i, $i \in I$. Let

$$\underline{J}(P, S) = \sum_{R_i \subset S^\circ} vol(R_i), \quad \overline{J}(P, S) = \sum_{R_i \cap \overline{S} \neq \emptyset} vol(R_i),$$

and define

$$\underline{c}(S) := \sup_P \{\underline{J}(P, S)\} \quad \overline{c}(S) := \inf_P \{\overline{J}(P, S)\}.$$

We say that S is *Jordan measurable of Jordan measure* $c(S) \in \mathbb{R}$ iff $\overline{c}(S) = \underline{c}(S) = c(S)$.

   (a) Show that a bounded subset $B \subset \mathbb{R}^n$ is Jordan measurable iff $c(\partial B) = 0$.

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(b) If B is Jordan measurable and $f : B \to \mathbb{R}$ a function, then f is integrable iff the discontinuities of f form a set of measure zero.

5. *(10 points)* Let $R = [0, a] \times [0, 2\pi] \subset \mathbb{R}^2$, and consider the function $T : R \to \mathbb{R}^2$ given by $T(r, \theta) = (x, y)$ where $x = r\cos\theta$ and $y = r\sin\theta$. Show that T maps the rectangle R onto the closed disc D with center $(0, 0)$ and radius a, and that T is one-to-one in the interior of R. Prove that for any continuous function f on D,

$$\int_D f(x,y)dxdy = \int_0^a \int_0^{2\pi} f(T(r,\theta))rdrd\theta.$$

(Note that the Change of Variables Formula as we have proved it does not apply directly.)

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