Calculus
Final exam
27/11/23, 9:30-12:30

You may use Prof. Ramadas' notes or your class notes. No other sources are permitted. In a multi-part question, the result of one part may be used in a succeeding part even if you have not been able to derive it.

1. ($2.5 \times 4 = 10$ *points*) True or false? Give either a proof or a counterexample to justify your answer.

   (a) Let f be a real-valued (not necessarily continuous) function defined on the rectangle $[0,1] \times [0,1]$. If both the iterated integrals $\int_0^1 \int_0^1 f dx dy$ and $\int_0^1 \int_0^1 f dy dx$ exist, then they must be equal.

   (b) If $U \subset \mathbb{R}^2$ is an open such that the line integral of any curl-free vector field on it along a closed curve is 0 then U is star-shaped.

   (c) The *determinant* alternating form *det* on $\mathbb{R}^n$ is defined as follows. Given n vectors $v_i = (v_{i,1}, \dots, v_{i,n})$, we have

   $$det(v_1, \dots, v_n) := det[v_{i,j}].$$

   Any alternating n form on $\mathbb{R}^n$ is a scalar multiple of *det*.

   (d) Let $f : U \to \mathbb{R}^n$ be a smooth 1-1 map, where $U \subset \mathbb{R}^n$ is an open subset. If $J_f(x) = 0$, $\forall x \in U$, then the image of f cannot contain an open set.

2. (*10 points*) Let $a > b > 0$ be real constants, and $T : [0,\pi] \times [0, 2\pi] \to \mathbb{R}^3$ the "half torus" given by

   $$(\theta, \psi) \mapsto ((a + b\cos\psi)\cos\theta, (a + b\cos\psi)\sin\theta, b\sin\psi).$$

   Calculate $\iint_T z dy \wedge dz$. Show directly (i.e. without using Stokes Theorem) that this is the same as the integral $\frac{1}{2}\int_C z^2 dy$, where $C = \partial T$.

3. (*10 points*) Let S bs a closed C^2-surface in $\mathbb{R}^3$.

   (a) Using cylindrical coordinates ($x = r\cos\theta, y = r\sin\theta, z = z$), show that $\iint_S z r dr \wedge d\theta$ is the volume of the region enclosed by S.

   (b) Calculate the volume of the solid $x^2 + y^2 \leq 1$, $0 \leq z \leq 4 - x^2 - y^2$.

4. (*10 points*)

   (a) Assume that ω is a 1-form in an open set $E \subset \mathbb{R}^n$ such that $\int_\gamma \omega = 0$ for every C'-closed curve γ in E. Prove that ω is *exact* in E, that is $\exists f \in C^\infty(E)$ such that $\omega = df$.

   (b) Assume ω is a closed 1-form in $\mathbb{R}^3 - \{0\}$. Prove that ω is exact.

5. (*10 points*) For $0 < x < \infty$, define the gamma function

   $$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt.$$

It satisfies $\Gamma(x+1) = x\Gamma(x)$, $\forall x \in (0, \infty)$. Prove the identity

$$\int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{1}$$

following the steps below:

(a) Define $(u, v) = T(s, t)$ on the strip $S = \{(s, t) : s \in (0, \infty), t \in (0, 1)\} \subset \mathbb{R}^2$ by setting $u = s - st$, $v = st$. Show that T is a 1-1 mapping of the strip onto the positive quadrant Q in $\mathbb{R}^2$. Show that $J_T(s, t) = s$.

(b) For $x > 0, y > 0$, let $f(u, v) = u^{x-1}e^{-u}v^{y-1}e^{-v}$. Assuming the change-of-variables formula for the mapping T and the function f, calculate the integral of f over Q by converting the integral to one over S, and hence derive the identity (1). It may be useful to know that for a nonnegative locally-integrable function ρ, the integral over a non-compact rectangle R is the supremum of $\int_S \rho$ over all compact rectangles S contained in R.

(c) Prove the change-of-variables formula for the mapping T and the function f.

6. (*10 points*)The the i-th *cohomology group* of an open subset $U \subset \mathbb{R}^n$ is defined as the quotient

$$H^i(U) := \frac{\ker(d : \Omega^i(U) \to \Omega^{i+1}(U))}{d(\Omega^{i-1})}.$$

(Note that the denominator is a subspace of the numerator as $d \circ d = 0$.) Calculate $H^1(\mathbb{R}^2 - \{0\})$ following the steps below.

(a) Let $\omega = f dx$ be a 1-form on $[0, 1]$ where $f(0) = f(1)$. Show that there is a unique number λ such that $\omega - \lambda dx = dg$ for some function satisfying $g(0) = g(1)$.
(Hint: Reverse-engineer by integrating $\omega - \lambda dx = dg$.)

(b) Let $\omega \in \Omega^1(\mathbb{R}^2 - \{0\})$ be a *closed* 1-form, that is, $d\omega = 0$. Show that $\omega = \lambda\eta + dg$, where $\eta = \frac{-ydx+xdy}{x^2+y^2}$, and $g \in C^\infty(\mathbb{R}^2 - \{0\})$.

(c) Show that $H^1(\mathbb{R}^2 - \{0\}) \cong \mathbb{R}$.