Analysis 2 Quiz 1 Answers (draft)

Write your answers on this paper. *Often this involves completing the given sentences.* You should not need more space but you may use an extra sheet if necessary. Do write your roll number on it. **Explain everything** (unless specified otherwise) but do so *succinctly* . You may use any standard theorem proved in class. Clearly specify which result you are using. You may use that for any metric space (X, d) and any $x \in X$, the functions $d(x, -)$ and $d(-, -)$ are continuous.

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1. Let V be a subset of a metric space (X, d). We defined V to be open if for each point p in V, there is an $r > 0$ such that $B_r(p)$ is contained in V.

(i) From this definition it is immediate that any union of open sets is open. We can also see that for open sets $V_1, \ldots, V_n$, the set $V = V_1 \cap \ldots \cap V_n$ is open. Reason: Fix an arbitrary point p in V. For each $i = 1, \ldots, n$, we have a positive real number r_i for p due to V_i being open. To show V is open we may take r in the definition of open set to be the smallest of all r_i.

(ii) Fix a point q in X and fix a positive real number s. Then $B_s(q)$ is an open set of X because, for any p in $B_s(q)$, we may take $r = s - d(p, q)$ (or any smaller positive number) because of triangle inequality: for this value of r and for any z in $B_r(p)$, we have $d(q, z) \leq d(q, p) + d(q, z) < d(q, p) + r \leq s$, showing $z \in B_s(q)$ and hence $B_r(p) \subset B_s(q)$.

(iii) Is the set of real numbers a metric space if we define $d(a, a) = 0$ and $d(a, b) = 2023$ for $a \neq b$? If not, state why. If yes (you do not need to prove it), describe precisely which subsets of $(\mathbb{R}, d)$ are open and why.

Yes (easy to check). All subsets of $(\mathbb{R}, d)$ are open under this metric because any singleton set $\{p\} = B_r(p)$ for any radius $r \leq 2023$ and a union of open sets is open.

2. (i) In a normed linear space V with norm $|| \ . \ ||$, "a sequence s_n converges to s" means the following: given any $\epsilon > 0$ there is a positive integer M such that, whenever n is $\geq M$, the following must be true for s_n: one must have $||s_n - s|| < \epsilon$. (Either of the inequalities can be replaced by a weak one.)

(ii) Claim: if a_n converges to a and b_n converges to b in $\mathbb{R}$ (under the usual absolute value norm), then the sequence $s_n = (a_n, b_n)$ converges to (a, b). Prove this using the definition in (i) using a suitable norm on $\mathbb{R}^2$.

Proof: Given $\epsilon > 0$, we have constants M_1, M_2 such that $|a_n - a| < \epsilon$ for $n > M_1$ and $|b_n - b| < \epsilon$ for $n > M_1$. Let $|| \ . \ ||$ be the ∞ norm (i.e., the max norm) on $\mathbb{R}^2$. For $n > M =$ the larger of M_1 and M_2, we have $||(a_n, b_n) - (a, b)|| = ||(a_n - a, b_n - b)|| = max(|a_n - a|, |b_n - b|) < \epsilon$.

(iii) Complete the sentence and justify. The claim in (ii) is true when the norm on $\mathbb{R}^2$ is taken to be any norm because all norms on a finite dimensional vector space are equivalent. Furthermore, if $s_n \to s$ wrt one norm then $s_n \to s$ wrt any equivalent norm. (Recall why this is true starting with the definition of equivalence.)

3. Let $T \subset S \subset$ the Euclidean metric space $\mathbb{R}^2$. If possible, give an example of the following. (Each part is independent of the others.) If the given situation is impossible, explain very briefly using standard facts.

(i) T is open in $\mathbb{R}^2$ but not in S.

Not possible. $S \cap$ (any open set in $\mathbb{R}^2$) is open in S. So if T is open in $\mathbb{R}^2$ then $S \cap T = T$ is open in S.

(ii) T is open in S but not in $\mathbb{R}^2$.

Let S be any subset of $\mathbb{R}^2$ that is not open in $\mathbb{R}^2$ (e.g., $S =$ a singleton). Let $T = S$. Other answers possible.

(iii) T is compact in S but not in $\mathbb{R}^2$.

Not possible. Compactness is intrinsic. See proof in class or Rudin 2.33.

(iv) S is closed in $\mathbb{R}^2$ and T is closed in S but T is not closed in $\mathbb{R}^2$.

Not possible. As T is closed in S, we have $T = S \cap C$ for some C closed in $\mathbb{R}^2$. As S too is closed in $\mathbb{R}^2$, so must be the intersection $S \cap C$ of two closed sets of $\mathbb{R}^2$.

(v) S is compact in $\mathbb{R}^2$ and has some limit point in $\mathbb{R}^2$ that is not in S.

Not possible. Being compact in the metric space $\mathbb{R}^2$, the set S must be closed in $\mathbb{R}^2$ and hence must contain all its limit points.

4. Consider the subset $A = \{(x, y) | x^{200} + y^{100} < 2023\}$ of $\mathbb{R}^2$. Visualize A and answer the following.

(i) A is open in $\mathbb{R}^2$ because, considering the <u>continuous</u> function $f(x, y) = x^{200} + y^{100}$ from $\mathbb{R}^2$ to $\mathbb{R}$, we observe that $A = f^{-1}$ of the open set $(-\infty, 2023)$.

(ii) Let g be a continuous function from A to $\mathbb{R}$. Prove that $g(A)$ must be bounded or give a counterexample.

Take $p = (a, b)$ where $a^{200} + b^{100} = 2023$, e.g. $a = 0, b = 2023^{\frac{1}{100}}$. For $q \in A$, define $g(q) = \frac{1}{d(p,q)}$.

(iii) Let h be a continuous function from $\mathbb{R}^2$ to $\mathbb{R}$. Prove that $h(A)$ must be bounded or give a counterexample.

$C = \{(x, y) | x^{200} + y^{100} \leq 2023\}$ contains A and is a closed and bounded subset of $\mathbb{R}^2$. (Why? Also see that $\overline{A} = C$). So C is compact by Heine-Borel. So $h(C)$ is compact. So $h(C)$ is a closed and bounded set in $\mathbb{R}$. In particular $h(A)$, being a subset of $h(C)$, is bounded.

(iv) Suppose for some sequence (x_n, y_n) of points in A, x_n converges to x and y_n converges to y in $\mathbb{R}$. Then $x^{200} + y^{100} \leq 2023$. True or false?

True. (x_n, y_n) converges to (x, y), so $f(x_n, y_n)$ converges $f(x, y)$. As each real number $f(x_n, y_n) < 2023$, the limit of this sequence of real numbers must be ≤ 2023. (This is from Analysis 1 and easy to see using ϵ-δ.)

(v) Suppose B is a subset of $\mathbb{R}^2$ that is disjoint from A. Prove the following claim or give a counterexample: there is a continuous function from $\mathbb{R}^2$ to $\mathbb{R}$ that takes constant value 1 on A and constant value 0 on B.

False, e.g., take $B = \{p\}$ with p as in the answer to (ii) above. Any continuous function that has constant value on A must have the same value on p.