Analysis 2 HW 1

1. Suppose a vector space V over $\mathbb{R}$ has norms $||\cdot||$ and $||\cdot||'$. Consider the following three statements.

(a) The two norms are equivalent, i.e., there are $a, b > 0$ such that for all $v \in V$, one has $a||v|| \leq ||v||' \leq b||v||$.
(b) A subset S of V is open under $||\cdot|| \Leftrightarrow S$ is open for $||\cdot||'$ (i.e., both norms induce the same topology).
(c) A sequence x_n in V converges under $||\cdot|| \Leftrightarrow$ it converges under $||\cdot||'$ and in that case the limit under each norm is the same.

Revise for yourself proofs of (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c), both done in class. Then show the reverse implications (b) $\Rightarrow$ (a) and (c) $\Rightarrow$ (a).

2. Let (X, d) be a metric space and let S be a subset of X, considered as a metric space under (the restriction of) the same distance function. In this exercise, every mention of some set being open (or closed) *must* be followed by the metric space in which it is so, e.g., "open in X" or "open in S".

(i) Carefully write down a full proof of the following result that was sketched in class. A subset T of S is open in the metric space S if an only if there exists a set V open in X such that $T = S \cap V$. Deduce that if S is open in X then (T is open in $S \Leftrightarrow T$ is open in X).

(ii) Write the analogous statements for closed sets and give a short clean justification.

3. (i) For a subset S of a metric space (X, d), show that the following two conditions are equivalent for $x \in X$. Both characterize when x is in $\overline{S}$ and we call such x adherent points of S.

(a) The intersection of S with every open ball centered at x contains at least one point.
(b) There is a sequence x_n in S converging to x.

Note: quickly see for yourself that replacing "open ball" in (a) by "neighborhood" is easily equivalent.

(ii) Formulate the analogues of (a) and (b) for x to be a *limit* point of S and prove their equivalence. In your proof you can just indicate what changes from the proof in (i) instead of repeating the entire argument.

Optional: By definition, we call a subset C of a metric space X *perfect* if C is closed and each point of C is a limit point of C. There are non-obvious perfect sets, e.g., it is possible for a perfect subset of $\mathbb{R}$ to contain no interval. Look up how the Cantor set is perfect.

(iii) Show that given a real $r > 0$ and a point c in a metric space X, the set $\{x | d(c, x) \leq r\}$ must be closed. Find a silly example showing that in a general metric space X this set need NOT be the closure of the open ball $\{x | d(c, x) < r\}$. What if X is the Euclidean space $\mathbb{R}^n$?

4. For metric spaces (X, d_X) and (Y, d_Y), the following conditions on a function $X \xrightarrow{f} Y$ are equivalent.

(a) For any $a \in X$ and any $\epsilon > 0$ there exists a $\delta > 0$ such that $d_X(x, a) < \delta \Rightarrow d_Y(f(x), f(a)) < \epsilon$.

(b) For any $a \in X$ and any sequence x_n converging to a, the sequence $f(x_n)$ converges to $f(a)$.

(c) f^{-1}(any open set of Y) is open in X.

(c') f^{-1}(any closed set of Y) is closed in X.

(d) For any subset S of X, $f(\overline{S}) \subset \overline{f(S)}$.

(d') A statement in terms of interiors that you should formulate.

We discussed the equivalence of (a) and (c) in detail and briefly discussed others. Write

(i) a proof of equivalence of (a) and (b),
(ii) a proof of equivalence of (c) or (c') and (d). This proof should not mention the metric at all.
(iii) the correct condition in (d') and a proof of its equivalence with (d).

Turn over →

5. (i) Product spaces. If (X_1, d_1) and (X_2, d_2) are metric spaces, their cartesian product $X_1 \times X_2$ becomes a metric space under the max metric. Show that open sets of $X_1 \times X_2$ under this metric are precisely unions of sets of the form (open ball of $X_1\times$ open ball of X_2). In the previous statement, can we replace "open ball" by arbitrary open set of the respective X_i? Are the two projection maps $\pi_i : X_1 \times X_2 \to X_i$ continuous?

(ii) For a fixed point p in a metric space X, observe that the real valued function $x \to d(p, x)$ is continuous. (See that this permits a 1-sentence proof of an earlier problem.) Show that under the topology on $X \times X$ from part (i), $(x, y) \to d(x, y)$ is continuous.

Optional: Suppose for each $i = 1, \dots, n$, we have a metric space X_i with distance function d_i. For any $p \in [1, \infty]$, construct a "combined p-metric built on top of the given d_i" on the product set $\Pi_{i=0}^n X_i$ and prove it is a metric. Are all such p-metrics "equivalent" under a suitable definition of equivalence? Look up some different notions of equivalence for metrics, e.g., topological (which is analogous to (b) in Q1), biLipschitz (analogous to (a) in Q1). For norms these notions give equivalent conditions as seen in Q1, but not for metrics. One can also use a similar construction for combining *norms* on normed linear spaces and get many norms by using this idea repeatedly, e.g., see the example of a "composite norm" at the wikipedia entry on "norm (mathematics)". All such norms on $\mathbb{R}^n$ are of course equivalent, as we saw in class.

Point set topology: basic definitions and optional exercises

By definition a topology on a set X consists of a subset $\mathcal{T}$ of the power set of X such $\mathcal{T}$ is closed under (arbitrary) unions and finite intersections. The elements of $\mathcal{T}$ are called open sets of X under $\mathcal{T}$. The complements of open sets are called closed. Often we just mention X and suppress $\mathcal{T}$ because it is understood.

Of course every metric space becomes a topological space, as we have seen. Thus we turned a *result* for metric spaces into a new *definition* that can be applied in a wider context. This is something of a theme. In a general topological space, there is no distance and there are no balls, but one can still define several familiar notions directly in terms of the prespecified collection of open sets. E.g., for any subset S of X the interior of S is defined to be the union of all open sets contained in S. Observe that the interior of S is the largest open set of X contained in S. Formulate for yourself the analogous definitions/statements for $\overline{S}$, interior points, adherent points and limit points purely in terms of open sets.

A map $f : X \to Y$ bewteen topological spaces is called continuous if f^{-1}(any open set of Y) is open in X. See Q4 above. We turned statement (c) there into a definition. (c) and (d) are equivalent for any topological space and your proof in Q4(ii) should work. The ϵ-δ definition does not even make sense now.

What about the sequence criterion (b) in Q4? It can still be formulated (how will you define limit of a sequence? is a limit unique?) but the statement is no longer equivalent to continuity. This is an example of another theme, namely sequences are not nearly so useful as they are for metric spaces. One can introduce two generalizations of the notion of a sequence – nets and filters – either of which can act as a replacement for sequences. Look this up if you care (but it is not necessary at all).

A. For a subset S of a topological space $(X, \mathcal{T})$, define the subspace topology $(S, \mathcal{T}')$. The result in Q2(i) is a guide. Show that $\mathcal{T}'$ is indeed a topology and the inclusion map from S to X is continuous.

B. Given topological spaces X_1, X_2, define a topology on the cartesian product $X_1 \times X_2$ by saying that a set is open if it is a union of sets of the form (open in X_1)$\times$ (open in X_2). (i) Check that this is a topology. Show quickly that the projection maps are continuous. (ii) A set map $f : Y \to X_1 \times X_2$ is given by $y \mapsto (f_1(y), f_2(y))$ where $f_i : Y \to X_i$ is a set map. If Y is a topological space, then show that f is continuous if and only if both component maps f_i are continuous.

Remark: The topology on $X_1 \times X_2$ is defined so as to make the result in (ii) true. Prove the following "universal property of products" and meditate on what kind of statement it is. We say that the two projections from $X_1 \times X_2$ to X_i are a universal configuration in the sense that every possible configuration of two continuous maps f_i from $Y \to X_i$ *factors uniquely through the universal configuration. As a result, the universal property characterizes the product topological space "up to unique isomorphism".* One has to decipher the italicized part. If you think this is mumbo jumbo, you are right. Ignore it. If you like it, be aware that this is "just" formalism with debatable mathematical value.