Analysis 2 Midterm

Begin each problem on a different sheet of paper. *If you wish*, upload this exam on moodle at the end so you will always have a copy.

Explain everything. You may appeal to (i) any standard result proved in class or in Analysis I as well as (ii) theoretical results proved in the required homework problems unless, of course, the question asks for the proof of such a result. In all cases where you are appealing to a result, you must clearly specify which result you are using and why it is applicable. If in doubt, ask me.

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1. (i) Let $p \in U \subset \mathbb{R}^n$ and let $f : U \to \mathbb{R}^k$ be a function. Define $f'(p)$. Include any additional hypotheses needed on the nature of U and/or f. Give a parallel definition of $f''(p)$.

In the remaining parts, you may appeal to any standard results with proper justification.

(ii) Now let $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (xy^2, x^2 + y^4)$ and let $p = (1, 1)$. Justify why $f'(p)$ exists and specify $f'(p)$ in terms of finitely many real numbers, stating the precise way in which these numbers describe $f'(p)$ as defined in part (i). Specify all points at which f is C^1.

(iii) Specify all points at which f is C^2. For $p = (1, 1)$, specify $f''(p)$ in terms of finitely many real numbers stating precisely how these numbers describe $f''(p)$ as defined in part (i). What is the minimum number of distinct calculations you need to do describe $f''(p)$?

2. (i) Let A be a subset of a metric space (K, d). Define what it means for a point $x \in K$ to be a limit point of A. Your definition should be in terms of neighborhoods of x. Suppose $|A| = 2023$. State all possible values of the number of limit points of A. State all possible values of the number of adherent points of A.

(ii) Prove that if A is infinite and if K is compact, then there must be some $x \in K$ that is a limit point of A. Possible beginning of a proof: Suppose there is no limit point. Then each $x \in K$ has an open neighborhood U_x such that ...

(iii) Given a sequence x_n in a compact metric space K, show how to extract a convergent subsequence x_{n_m}.

(iv) Give an example of an infinite subset A of bounded set K in $\mathbb{R}^n$ for which the conclusion of (ii) fails, i.e., such that A has no limit point in K. Is it necessary that such a set A must have a limit point in $\mathbb{R}^n$?

3. A function $f : (X, d_X) \to (Y, d_Y)$ between metric spaces is called *uniformly continuous* if for every pair of points x_1, x_2 in X the following is true: ... (complete the sentence). Using problem 2 or otherwise show that if f is continuous and X is compact, then (i) $f(X)$ is compact and (ii) f is uniformly continuous.

4. (i) For functions $g : \mathbb{R}^a \to \mathbb{R}^b$ and $f : \mathbb{R}^b \to \mathbb{R}^c$, carefully state the chain rule at $p \in \mathbb{R}^a$. Be sure to include the precise hypotheses.

(ii) For differentiable *real* valued functions $g : \mathbb{R}^n \to \mathbb{R}$ and $f : \mathbb{R}^n \to \mathbb{R}$, show that the functions $\frac{1}{g} : x \mapsto \frac{1}{g(x)}$ and $fg : x \mapsto f(x)g(x)$ are differentiable wherever defined and calculate the derivative of each function at any given $p \in \mathbb{R}^n$. Hints: Keep mind the answers from single variable calculus. Why are there two parts to this problem? In return for hints, I expect utter theoretical clarity in your presentation.

Turn over!

5. Short independent problems. Do as many as you can.

A. Consider a function $f : \mathbb{R}^n \to \mathbb{R}^k$ given by $f(x) = (f_1(x), \ldots, f_k(x))$ where $f_1, \ldots, f_k$ are k real valued functions. If $n = k$ and f is identity, then we call each f_i a projection and denote it by π_i. Note that π_i are linear and continuous. Using this setup as appropriate, answer the following for general f.

(i) If f is continuous, then so is each f_i because ... (give a very short answer). Does the same statement hold when "continuous" is replaced by "differentiable"? By "C^1"?

(ii) Is the converse of each of the three statement considered in (i) true? Justify each answer briefly but precisely.

B. Give a necessary and sufficient topological condition on subsets A and B of $\mathbb{R}^n$ for there to exist a continuous function $\mathbb{R}^n \to \mathbb{R}$ that takes constant value 1 on A and constant value 0 on B.

C. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $(0,0) \mapsto 0$ and otherwise $(x,y) \mapsto \frac{xy^2}{x^2+y^4}$. Is f differentiable at the origin? Is it continuous at the origin?

D. Find all differentiable functions $f : \{(x,y) \in \mathbb{R}^2 | 2022 < x^2 + y^2 < 2023\} \to \mathbb{R}$ whose derivative is identically 0.

E. Show from first principles that a bilinear map $\mathbb{R}^n \times \mathbb{R}^m \xrightarrow{\otimes} \mathbb{R}^\ell$ (denoted $(x,y) \mapsto x \otimes y$) is differentiable and find its derivative at (v,w), evaluated at (h,k). Be sure to justify all steps.

F. Fix a positive integer $n > 1$. Is there a point in $\mathbb{R}^n$ at which every norm is necessarily differentiable? Is there a point in $\mathbb{R}^n$ at which every norm is necessarily non-differentiable?