Analysis II HW 4

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Problem 1:

Suppose f is a real function on $[a,b]$, n is a positive integer, and f^{n-1} exists for every $t \in [a,b]$. Let α, β, and P be as in Taylor's Theorem (5.15). Define

$$Q(t) = \frac{f(t) - f(\beta)}{t - \beta}$$

for $t \in [a,b], t \neq \beta$, differentiate

$$f(t) - f(\beta) = (t - \beta)Q(t)$$

$n - 1$ times at $t = \alpha$, and derive the following version of Taylor's theorem:

$$f(\beta) = P(\beta) + \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta - \alpha)^n.$$

Solution 1:

We shall prove using induction that for $1 \leq k \leq n - 1$,

$$f^{(k)}(t) = (t - \beta)Q^{(k)}(t) + kQ^{(k-1)}(t). \tag{1}$$

Clearly for $k = 1$, this is true because

$$f'(t) = (t - \beta)Q'(t) + Q(t).$$

Assume that for some $1 \leq k < n - 1$,

$$f^{(k)}(t) = (t - \beta)Q^{(k)}(t) + kQ^{(k-1)}(t).$$

Then

$$\begin{aligned} f^{(k+1)}(t) &= \frac{\mathrm{d}}{\mathrm{d}t} f^{(k)}(t) \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \left\{ (t - \beta)Q^{(k)}(t) + kQ^{(k-1)}(t) \right\} \\ &= (t - \beta)Q^{(k+1)}(t) + Q^{(k)}(t) + kQ^{(k)}(t) \\ &= (t - \beta)Q^{(k+1)}(t) + (k+1)Q^{(k)}(t) \end{aligned}$$

and hence we have proved that equation (1) holds.

Now, we have,

$$\begin{aligned} \frac{f^k(\alpha)}{k!}(\beta - \alpha)^k &= \frac{1}{k!} \left\{ (\alpha - \beta)Q^{(k)}(t) + kQ^{(k-1)}(\alpha) \right\} (\beta - \alpha)^k \\ &= -\frac{(\beta - \alpha)^{k+1}}{k!} Q^{(k)}(t) + \frac{(\beta - \alpha)^k}{(k-1)!} Q^{(k-1)}(\alpha). \end{aligned}$$

Since the sum telescopes, therefore,

$$P(\beta) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(\beta - \alpha)^k = f(\beta) - \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta - \alpha)^n,$$

and hence,

$$f(\beta) = P(\beta) + \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta - \alpha)^n.$$

■

Problem 2:

Let $\mathbf{f} = (f_1, f_2)$ be mapping of $\mathbb{R}^2$ into $\mathbb{R}^2$ given by

$$f_1(x, y) = e^x \cos y, \quad f_2(x, y) = e^x \sin y.$$

(a) What is the range of $\mathbf{f}$?
(b) Show that the Jacobian of $\mathbf{f}$ is not zero at any point of $\mathbb{R}^2$. Thus every point of $\mathbb{R}^2$ has a neighborhood in which $\mathbf{f}$ is one-to-one. Nevertheless, $\mathbf{f}$ is not one-to-one on $\mathbb{R}^2$.
(c) Put $\mathbf{a} = (0, \pi/3)$, $\mathbf{b} = \mathbf{f}(\mathbf{a})$, let $\mathbf{g}$ be the continuous inverse of $\mathbf{f}$, defined in a neighborhood of $\mathbf{b}$, such that $\mathbf{g}(\mathbf{b}) = \mathbf{a}$. Find an explicit formula for $\mathbf{g}$, compute $\mathbf{f}'(\mathbf{a})$ and $\mathbf{g}'(\mathbf{b})$, and verify the formula $\mathbf{g}'(\mathbf{b}) = \{\mathbf{f}'(\mathbf{g}(\mathbf{b}))\}^{-1}$ (d) What are the images under $\mathbf{f}$ of lines parallel to the coordinate axes?

Solution 2:

(a) If $(w, z) \neq 0$, choose y so that

$$\cos y = \frac{w}{\sqrt{w^2 + z^2}} \quad \text{and} \quad \sin y = \frac{z}{\sqrt{w^2 + z^2}}.$$

Let $x = \ln \sqrt{w^2 + z^2}$ so that $e^x = \sqrt{w^2 + z^2}$. Then,

$$w = e^x \cos y = f_1(x, y) \text{ and } z = e^x \sin y = f_2(x, y).$$

Therefore, every point except $(0, 0)$ is in the range of $\mathbf{f}$ because for any point $(w, z) = \mathbf{f}(x, y)$, we have $w^2 + z^2 = (e^x)^2 > 0$, i.e., the range of $\mathbf{f}$ is $\mathbb{R}^2 \setminus (0, 0)$.
(b) We have,

$$\mathbf{f}'(x, y) = \begin{pmatrix} D_1 f_1(x, y) & D_1 f_2(x, y) \\ D_2 f_1(x, y) & D_2 f_2(x, y) \end{pmatrix} = \begin{pmatrix} e^x \cos y & e^x \sin y \\ -e^x \sin y & e^x \cos y \end{pmatrix}.$$

Therefore, the Jacobian of $\mathbf{f}$ at (x, y) is given by

$$J_{\mathbf{f}}(x, y)) = \det[\mathbf{f}'(x, y)] = e^{2x}(\cos^2 y + \sin^2 y) = e^{2x} \neq 0$$

for all x. By the inverse function theorem, this implies that $\mathbf{f}$ is locally invertible at every point of $\mathbb{R}^2$. In other words, every point in $\mathbb{R}^2$ has a neighborhood in which $\mathbf{f}$ is one-to-one. However, since $\mathbf{f}(x, y) = \mathbf{f}(x, y + 2n\pi)$ for all $n \in \mathbb{Z}$, it follows that $\mathbf{f}$ is not one-to one.
(c) Let $w = f_1(x, y) = e^x \cos y$, $z = f_2(x, y) = e^x \sin y$. Then in a neighborhood of $\mathbf{b} = \mathbf{f}(\mathbf{a}) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ where $w \neq 0$, we have $w^2 + z^2 = e^{2x}$ and $\frac{z}{w} = \tan y$. Thus, we have $\mathbf{g}(w, z) = \left(\ln \sqrt{w^2 + z^2}, \arctan(\frac{z}{w})\right)$. We then have,

$$\mathbf{f}'(x, y) = \begin{pmatrix} e^x \cos y & e^x \sin y \\ -e^x \sin y & e^x \cos y \end{pmatrix} \quad \text{and} \quad \mathbf{g}'(w, z) = \begin{pmatrix} \frac{w}{w^2+z^2} & \frac{-z}{w^2+z^2} \\ \frac{z}{w^2+z^2} & \frac{w}{w^2+z^2} \end{pmatrix}.$$

Therefore,

$$\mathbf{f}'(\mathbf{a})\mathbf{g}'(\mathbf{b}) = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and hence,

$$\mathbf{g}'(\mathbf{b}) = \{\mathbf{f}'(\mathbf{a})\}^{-1} = \{\mathbf{f}'(\mathbf{g}(\mathbf{b}))\}^{-1}.$$

(d) Lines parallel to the x-axis are of the form (t, a), where $t \in \mathbb{R}$ is the parameter and a is a constant. Under $\mathbf{f}$, these points are mapped to

$$\mathbf{f}(t, a) = (f_1(t, a), f_2(t, a)) = (e^t \cos a, e^t \sin a),$$

which is a point on the circle of radius e^t centered at the origin. Therefore, the image of any line parallel to the x-axis is a circle centered at the origin.

Similarly, lines parallel to the y-axis are of the form (a, t), and under $\mathbf{f}$ these points are mapped to

$$\mathbf{f}(a, t) = (f_1(a, t), f_2(a, t)) = (e^a \cos t, e^a \sin t),$$

which is also a point on a circle centered at the origin. Therefore, the image of any line parallel to the y-axis is also a circle centered at the origin.

Therefore, under $\mathbf{f}$, lines parallel to the coordinate axes are mapped to circles centered at the origin. ■

Problem 3:

Show that the system of equations

$$\begin{aligned} 3x + y - z + u^2 &= 0 \\ x - y + 2z + u &= 0 \\ 2x + 2y - 3z + 2u &= 0 \end{aligned}$$

can be solved for x, y, u in terms of z; for x, z, u in terms of y; for y, z, u in terms of x; but not for x, y, z in terms of u.

Solution 3:

Consider the transformation

$$\begin{aligned} \mathbf{f}(x, y, z, u) &= (3x + y - z + u^2, x - y + 2z + u, 2x + 2y - 3z + 2u) \\ &= (f_1(x, y, z, u), f_2(x, y, z, u), f_3(x, y, z, u)) \text{ (say)}. \end{aligned}$$

Then the matrix of the derivative of the transformation $\mathbf{f}$ is given by

$$\mathbf{f}'(x, y, z, w) = (D_j f_i(x, y, z, w)) = \begin{pmatrix} 3 & 1 & -1 & 2u \\ 1 & -1 & 2 & 1 \\ 2 & 2 & -3 & 2 \end{pmatrix}.$$

Note that $\mathbf{f}(\mathbf{0}) = \mathbf{0}$. The determinant of the x, y, u part of the matrix is

$$\det \begin{pmatrix} 3 & 1 & 2u \\ 1 & -1 & 1 \\ 2 & 2 & 2 \end{pmatrix} = 3(-2 - 2) - (2 - 2) + 2u(2 + 2) = 8u - 12$$

which is not equal to $\mathbf{0}$ near $\mathbf{0}$, so by the implicit function theorem, there is a solution to $\mathbf{f}(x(z), y(z), z, u(z)) = \mathbf{0}$ near $\mathbf{0}$.

The determinant of the x, z, u part of the matrix is

$$\det \begin{pmatrix} 3 & -1 & 2u \\ 1 & 2 & 1 \\ 2 & -3 & 2 \end{pmatrix} = 3(4+2) + (2-2) + 2u(-3-4) = 21 - 14u$$

which is not equal to $\mathbf{0}$ near $\mathbf{0}$, so by the implicit function theorem, there is a solution to $\mathbf{f}(x(y), y, z(y), u(y)) = \mathbf{0}$ near $\mathbf{0}$.

Similarly, the determinant of the y, z, u part of the matrix is

$$\det \begin{pmatrix} 1 & -1 & 2u \\ -1 & 2 & 1 \\ 2 & -3 & 2 \end{pmatrix} = (4+3) + (-2-2) + 2u(3-4) = 3 - 2u$$

which is not equal to $\mathbf{0}$ near $\mathbf{0}$, so by the implicit function theorem, there is a solution to $\mathbf{f}(x, y(x), z(x), u(x)) = \mathbf{0}$ near $\mathbf{0}$.

However, the determinant of the x, y, z part of the matrix is

$$\det \begin{pmatrix} 3 & 1 & -1 \\ 1 & -1 & 2 \\ 2 & 2 & -3 \end{pmatrix} = 3(3-4) - (-3-4) - (2+2) = -3 + 7 - 4 = 0,$$

and therefore the implicit function theorem cannot be applied. ■

Problem 4:

Define f in $\mathbb{R}^3$ by

$$f(x, y_1, y_2) = x^2 y_1 + e^x + y_2.$$

Show that $f(0, 1, -1) = 0, (D_1 f)(0, 1, -1) \neq 0$, and that there exists therefore a differentiable function g in some neighborhood of $(1, -1)$ in $\mathbb{R}^2$, such that $g(1, -1) = 0$ and

$$f(g(y_1, y_2), y_1, y_2) = 0.$$

Find $(D_1 g)(1, -1)$ and $(D_2 g)(1, -1)$.

Solution 4:

We have, $f(0, 1, -1) = 0^2 \cdot 1 + e^0 + (-1) = 0$, and since

$$D_1 f(x, y_1, y_2) = \frac{\partial}{\partial x}(x^2 y_1 + e^x + y_2) = 2x y_1 + e^x,$$

so $D_1 f(0, 1, -1) = 2 \cdot 0 \cdot 1 + e^0 = 1 \neq 0$. Therefore, by the implicit function theorem, there exists a differentiable function g in some neighborhood of $(1, -1)$ in $\mathbb{R}^2$, such that $g(1, -1) = 0$ and $f(g(y_1, y_2), y_1, y_2) = 0$. Furthermore, since $D_2 f(0, 1, -1) = 0$ and $D_3 f(0, 1, -1) = 1$, so with $m = 1, n = 2$ in the implicit function theorem, we have

$$A_x = \begin{pmatrix} 1 \end{pmatrix} \quad \text{and} \quad A_y = \begin{pmatrix} 0 & 1 \end{pmatrix},$$

and the derivative of g at $(1, -1)$ is given by

$$g'(1, -1) = -A_x^{-1} A_y = -\begin{pmatrix} 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \end{pmatrix}.$$

Therefore, $D_1 g(1, -1) = 0$ and $D_2 g(1, -1) = -1$. ■

Problem 5:

Show that tangent vectors can be realized as velocity vectors of curves. More precisely, let U be an open set in $\mathbb{R}^n$. Let g be a C^1 map $U \to \mathbb{R}^m$. Let c a point in the image of g, $M = g^{-1}(c)$ and $p \in M$ such that $g'(p)$ is surjective. Recall that $T_pM =$ the kernel of $g'(p)$ is called the tangent space of M at p. Show that this tangent space is spanned by the velocity vectors of all C^1 paths γ in M based at p, i.e., by $\gamma'(0)$, where $\gamma : (-\epsilon, \epsilon) \to M$ is a C^1 function with $\gamma(0) = p$. Hint: use implicit function theorem to get the desired curve in the dependent coordinate.

Solution 5:

Let $\gamma : (-\epsilon, \epsilon) \to M$ be a C^1 path based at p, i.e., $\gamma(0) = p$. Define $F : (-\epsilon, \epsilon) \to \mathbb{R}^n$ by $F(t) = g(\gamma(t))$. Then $F(0) = c$ and $F'(0) = g'(\gamma(0))\gamma'(0) = g'(p)\gamma'(0) = 0$ since $\gamma'(0) \in T_pM$. By the implicit function theorem, there exists a C^1 function $\phi : (-\epsilon, \epsilon) \to \mathbb{R}^{n-m}$ such that $F(t) = c$ if and only if $\gamma(t) = \phi(t)$. Moreover, we have $\phi(0) = 0$ and $\phi'(0) = -(DF|_p)^{-1} g'(p)\gamma'(0)$, where $DF|_p$ denotes the derivative of F at $t = 0$. Since $g'(p)$ is surjective, $DF|_p$ is injective, so $(DF|_p)^{-1}$ exists. Therefore, $\phi'(0)$ is uniquely determined by $\gamma'(0)$, and we can express $\gamma'(0)$ in terms of $\phi'(0)$ as follows:

$$\gamma'(0) = -g'(p)^{-1}(DF|_p)\phi'(0).$$

Since $\phi'(0) \in \mathbb{R}^{n-m}$, we see that $\gamma'(0) \in T_pM$. Thus, every C^1 path γ in M based at p gives rise to a tangent vector $\gamma'(0)$, and the tangent space T_pM is spanned by the velocity vectors of all such paths.

Conversely, let $v \in T_pM$ be a tangent vector at $p \in M$. By definition of the tangent space, there exists a C^1 path $\gamma : (-\epsilon, \epsilon) \to M$ based at p such that $\gamma'(0) = v$. Let $\phi(t) = tv$ for $t \in (-\epsilon, \epsilon)$. Then $\phi(0) = 0$ and $\phi'(0) = v$. We want to find a C^1 path $\tilde{\gamma}$ in M based at p such that $\tilde{\gamma}'(0) = v$. By the implicit function theorem, we can solve for $\tilde{\gamma}$ by setting

$$F(t, x) = \gamma(t) - x = 0$$

and taking $x = \tilde{\gamma}(t)$. Then

$$\frac{d}{dt}F(t, \tilde{\gamma}(t)) = \gamma'(t) - \tilde{\gamma}'(t) = 0,$$

so $\tilde{\gamma}'(0) = \gamma'(0) = v$, as desired. Therefore, we have shown that the tangent space T_pM is indeed spanned by the velocity vectors of C^1 paths in M based at p. ■