Analysis II HW 2

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Problem 1:

If $f(0,0) = 0$ and

$$f(x,y) = \frac{xy}{x^2+y^2} \quad \text{if} \quad (x,y) \neq (0,0)$$

Prove that $(D_1 f)(x,y)$ and $(D_2 f)(x,y)$ exist at every point of $\mathbb{R}^2$, although f is not continuous at $(0,0)$.

Solution 1:

For $(x,y) \neq (0,0)$, we have

$$(D_1 f)(x,y) = \frac{\partial f(x,y)}{\partial x} = \frac{(x^2+y^2)y - (xy)(2x)}{(x^2+y^2)^2} = \frac{y(y^2-x^2)}{(x^2+y^2)^2}$$

and similarly,

$$(D_2 f)(x,y) = \frac{\partial f(x,y)}{\partial y} = \frac{(x^2+y^2)x - (xy)(2y)}{(x^2+y^2)^2} = \frac{x(x^2-y^2)}{(x^2+y^2)^2}$$

For $(x,y) = (0,0)$, we have,

$$(D_1 f)(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = 0$$

and similarly,

$$(D_2 f)(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = 0$$

i.e., the partial derivatives $(D_1 f)(x,y)$ and $(D_2 f)(x,y)$ exist at every $(x,y) \in \mathbb{R}^2$. However, if $\mathbf{x}_n = (\frac{1}{n}, \frac{1}{n})$, then $\mathbf{x}_n \to (0,0)$ as $n \to \infty$ but,

$$\lim_{n\to\infty} f(\mathbf{x}_n) = \lim_{n\to\infty} f\left(\frac{1}{n}, \frac{1}{n}\right) = \lim_{n\to\infty} \frac{1/n^2}{2/n^2} = \frac{1}{2} \neq (0,0)$$

and therefore, f is not continuous at $(0,0)$. ■

Problem 2:

Suppose that f is a real-valued function defined in an open set $E \subset \mathbb{R}^n$, and that the partial derivatives $D_1 f_1, \ldots, D_n f_n$ are bounded in E. Prove that f is continuous in E.

Solution 2:

To prove that f is continuous in E, we need to show that for any point $x \in E$ and any $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(\mathbf{y}) - f(\mathbf{x})| < \epsilon$ whenever $\mathbf{y} \in E$ and $|\mathbf{y} - \mathbf{x}| < \delta$.

Fix a point $\mathbf{x} \in E$ and $\epsilon > 0$. Since $D_i f_j$ are bounded in E, we have $|D_i f_j| \leq M_i$ for $1 \leq i, j \leq n$. Let $\mathbf{h} = \sum_{j=1}^n h_j \mathbf{e}_j$ be a vector in $\mathbb{R}^n$ with $|\mathbf{h}| < \delta$, where $\delta = \frac{\epsilon}{nM}$ and $M = \max_{1\leq i\leq n} M_i$. Put $\mathbf{v}_0 = \mathbf{0}$ and $\mathbf{v}_k = \sum_{i=1}^k h_i \mathbf{e}_i$. Then we have

$$f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) = \sum_{j=1}^n [f(\mathbf{x}+\mathbf{v}_j) - f(\mathbf{x}+\mathbf{v}_{j-1})]$$

Since $\mathbf{v}_j = \mathbf{v}_{j-1} + h_j\mathbf{e}_j$, the mean value theorem gives

$$f(\mathbf{x}+\mathbf{v}_j) - f(\mathbf{x}+\mathbf{v}_{j-1}) = h_j D_j f(\mathbf{x}+\mathbf{v}_{j-1}+\theta_j h_j \mathbf{e}_j)$$

for some $\theta_j \in (0,1)$. Therefore,

$$\begin{aligned}
|f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x})| &= \left|\sum_{j=1}^{n}[f(\mathbf{x}+\mathbf{v}_j) - f(\mathbf{x}+\mathbf{v}_{j-1})]\right| \\
&\leq \sum_{j=1}^{n} |f(\mathbf{x}+\mathbf{v}_j) - f(\mathbf{x}+\mathbf{v}_{j-1})| \\
&= \sum_{j=1}^{n} |h_j(D_j f)(\mathbf{x}+\mathbf{v}_{j-1}+\theta_j h_j\mathbf{e}_j)| \\
&\leq M\sum_{j=1}^{n} |h_j| \\
&\leq Mn|\mathbf{h}| < \epsilon
\end{aligned}$$

This implies that f is continuous in $E \subset \mathbb{R}^n$. ■

Problem 3:

Suppose that f is a differentiable real function in an open set $E \subset \mathbb{R}^n$, and that f has a local maximum at a point $\mathbf{x} \in E$. Prove that $f'(\mathbf{x}) = 0$.

Solution 3:

Suppose that f is a differentiable real function in an open set $E \subset \mathbb{R}^n$, and that f has a local maximum at a point $\mathbf{x} \in E$. We want to show that $f'(\mathbf{x}) = \mathbf{0}$.

By the definition of a local maximum, there exists a ball $B_r(\mathbf{x}) \subset E$ around $\mathbf{x}$ such that $f(\mathbf{x}) \geq f(\mathbf{y})$ for all $\mathbf{y} \in B_r(\mathbf{x}) \subset E$. Consider the function $g(t) = f(\mathbf{x}+t\mathbf{e}_j)$ for $1 \leq j \leq n$, where $\mathbf{e}_j$ is the j^{th} standard basis vector in $\mathbb{R}^n$. Since $B_r(\mathbf{x}) \subset E$, we have $\mathbf{x}+t\mathbf{e}_j \in B_r(\mathbf{x})$ for all t such that $|t| < r$. Thus, $g(t) \leq f(\mathbf{x})$ for all t such that $|t| < r$.

By the definition of the derivative, we have

$$D_j f(\mathbf{x}) = \lim_{t\to 0}\frac{f(\mathbf{x}+t\mathbf{e}_j) - f(\mathbf{x})}{t}.$$

Using the function g defined above, we have

$$D_j f(\mathbf{x}) = \lim_{t\to 0}\frac{g(t) - g(0)}{t}.$$

Since $g(t) \leq f(\mathbf{x})$ for all t such that $|t| < r$, we have

$$\frac{g(t)-g(0)}{t} \leq \frac{f(\mathbf{x}) - f(\mathbf{x}+t\mathbf{e}_j)}{t} \leq 0$$

for all t such that $|t| < r$. Taking the limit as $t \to 0$, we have

$$D_j f(\mathbf{x}) \leq 0.$$

Similarly, we can show that $D_j f(\mathbf{x}) \geq 0$ by considering the function $h(t) = f(\mathbf{x} - t\mathbf{e}_j)$ and using the same argument.

Thus, we have $D_j f(\mathbf{x}) = 0$ for $1 \leq j \leq n$. It follows by Theorem 9.17 that $f'(\mathbf{x}) = 0$. ■

Problem 4:

If f is a real function defined in a convex open set $E \subset \mathbb{R}^n$, such that $(D_1 f)(\mathbf{x}) = 0$ for every $\mathbf{x} \in E$, prove that $f(\mathbf{x})$ depends only on $x_2, \dots, x_n$.

Show that the convexity of E can be replaced by a weaker condition, but that some condition is required. For example, if $n = 2$ and E is shaped like a horseshoe, the statement may be false.

Solution 4:

Let $\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n) \in E$ be two points such that $x_i = y_i$ for $2 \le i \le n$. Consider the function $g : [0,1] \to E$ defined by $g(t) = (tx_1 + (1-t)y_1, x_2, \dots, x_n)$. Since E is convex, $g(t) \in E$ for every $t \in [0,1]$.

By the chain rule, we have

$$\begin{aligned}
\frac{d}{dt} f(g(t)) &= \nabla f(g(t)) \cdot \frac{d}{dt} g(t) \\
&= (D_1 f)(g(t)) \frac{d}{dt}(tx_1 + (1-t)y_1) + \sum_{j=2}^{n} (D_j f)(g(t)) \frac{d}{dt} x_j \\
&= (D_1 f)(g(t))(x_1 - y_1),
\end{aligned}$$

where we have used the fact that $(D_1 f)(\mathbf{x}) = 0$ for every $\mathbf{x} \in E$ and that $\frac{d}{dt} x_j = \frac{d}{dt} y_j = 0$ for $2 \le j \le n$.

Since f is defined in a convex set, the line segment connecting $\mathbf{x}$ and $\mathbf{y}$ is contained in E. Hence, we can apply the mean value theorem to the function $t \mapsto f(g(t))$ to get

$$f(\mathbf{x}) - f(\mathbf{y}) = f(g(1)) - f(g(0)) = \frac{d}{dt} f(g(t_0))(1-0) = (D_1 f)(g(t_0))(x_1 - y_1),$$

for some $t_0 \in (0,1)$. But $(D_1 f)(g(t_0)) = 0$, so $f(\mathbf{x}) - f(\mathbf{y}) = 0$. Therefore, f has the same value at any two points in E that have the same values for $x_2, \dots, x_n$. Hence, f depends only on $x_2, \dots, x_n$.

The convexity of E can be replaced by the weaker condition that E is connected. However, some condition is required, as the statement is false if $n = 2$ and E is shaped like a horseshoe. In this case, E is connected but f can have different values at points with the same values for x_2. For example, consider the function $f(x,y) = x^2 - y^2$ defined on the horseshoe-shaped set $E = (x,y) \in \mathbb{R}^2 : x^2 - y^2 < 1, x \neq 0$. At any point $(x,y) \in E$ with $x \neq 0$, we have $(D_1 f)(x,y) = 2x \neq 0$, but f depends on both x and y. ■

Problem 5:

Put $f(0,0) = 0$, and

$$f(x,y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}$$

if $(x,y) \neq (0,0)$. Prove that

(a) f, $D_1 f$ and $D_2 f$ are continuous in $\mathbb{R}^2$;

(b) $D_{12} f$ and $D_{21} f$ exist at every point of $\mathbb{R}^2$, and are continuous everywhere except at $(0,0)$;

(c) $(D_{12} f)(0,0) = 1$, and $(D_{21} f)(0,0) = -1$.

Solution 5:

(a) Clearly, f is continuous at every point $(x, y) \in \mathbb{R}^2$ other than $(0, 0)$. Since

$$|xy| \leq \frac{x^2 + y^2}{2},$$

so for $(x, y) \neq (0, 0)$, we have

$$|f(x, y)| = \frac{xy(x^2 - y^2)}{x^2 + y^2} \leq \frac{|xy| \cdot |x^2 - y^2|}{x^2 + y^2} \leq \frac{1}{2}|x^2 - y^2|.$$

Therefore,

$$\lim_{(x,y)\to(0,0)} |f(x, y)| \leq \lim_{(x,y)\to(0,0)} \left(\frac{1}{2}|x^2 - y^2|\right) = 0$$

and hence,

$$\lim_{(x,y)\to(0,0)} f(x, y) = 0 = f(0, 0)$$

i.e., f is continuous at $(0, 0)$. Therefore, f is continuous in $\mathbb{R}^2$.

For $(x, y) \neq (0, 0)$, we have

$$D_1 f(x, y) = \frac{\partial f(x, y)}{\partial x} = \frac{(x^2 + y^2) \cdot y(3x^2 - y^2) - xy(x^2 - y^2) \cdot 2x}{(x^2 + y^2)^2} = \frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$$

and,

$$D_2 f(x, y) = \frac{\partial f(x, y)}{\partial y} = \frac{(x^2 + y^2) \cdot x(x^2 - 3y^2) - xy(x^2 - y^2) \cdot 2y}{(x^2 + y^2)^2} = \frac{x(x^4 - 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$$

which are clearly continuous for $(x, y) \neq (0, 0)$.

Therefore,

$$\begin{aligned}
|D_1 f(x, y)| &= \frac{|y| \cdot |x^4 + 4x^2y^2 - y^4|}{(x^2 + y^2)^2} \\
&\leq |y| \left(\frac{|x^4 - y^4|}{(x^2 + y^2)^2} + \frac{4x^2y^2}{(x^2 + y^2)^2}\right) \\
&= |y| \left(\frac{|x^2 - y^2|}{x^2 + y^2} + \frac{4x^2y^2}{(x^2 + y^2)^2}\right) \\
&\leq |y|(1 + 1) = 2|y|
\end{aligned}$$

and similarly,

$$|D_2 f(x, y)| \leq 2|x|$$

Now, for $(x, y) = (0, 0)$, we have

$$D_1 f(0, 0) = \lim_{h\to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h\to 0} \frac{0 - 0}{h} = 0$$

and

$$D_2 f(0, 0) = \lim_{h\to 0} \frac{f(0, h) - f(0, 0)}{h} = \lim_{h\to 0} \frac{0 - 0}{h} = 0$$

Therefore,

$$\lim_{(x,y)\to(0,0)} |D_1 f(x,y)| \le \lim_{(x,y)\to(0,0)} (2|x|) = 0$$

and,

$$\lim_{(x,y)\to(0,0)} |D_2 f(x,y)| \le \lim_{(x,y)\to(0,0)} (2|y|) = 0.$$

This gives

$$\lim_{(x,y)\to(0,0)} D_1 f(x,y) = 0 = D_1 f(0,0)$$

and

$$\lim_{(x,y)\to(0,0)} D_2 f(x,y) = 0 = D_2 f(0,0)$$

i.e., $D_1 f$ and $D_2 f$ are also continuous at $(0,0)$. Therefore, $D_1 f$ and $D_2 f$ are continuous in $\mathbb{R}^2$.

(b) For $(x,y) \neq (0,0)$, we have

$$\begin{aligned} D_{12} f(x,y) &= D_1 D_2 f(x,y) \\ &= \frac{\partial}{\partial x}\left[\frac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2}\right] \\ &= \frac{(x^2+y^2)^2(5x^4 - 12x^2y^2 - y^4) - (x^5 - 4x^3y^2 - xy^4)\cdot 2(x^2+y^2)\cdot 2x}{(x^2+y^2)^4} \\ &= \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3} \end{aligned}$$

and similarly,

$$\begin{aligned} D_{21} f(x,y) &= D_2 D_1 f(x,y) \\ &= \frac{\partial}{\partial y}\left[\frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2+y^2)^2}\right] \\ &= \frac{(x^2+y^2)^2(x^4 + 12x^2y^2 - 5y^4) - (x^4y + 4x^2y^3 - y^5)\cdot 2(x^2+y^2)\cdot 2y}{(x^2+y^2)^2} \\ &= \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3} \end{aligned}$$

which are clearly continuous for $(x,y) \neq (0,0)$. We have,

$$D_{12} f(0,0) = \lim_{h\to 0} \frac{D_2 f(h,0) - D_2 f(0,0)}{h} = \lim_{h\to 0} \frac{h^4}{h^4} = 1$$

and

$$D_{21} f(0,0) = \lim_{h\to 0} \frac{D_1 f(0,h) - D_1 f(0,0)}{h} = \lim_{h\to 0} \frac{-h^4}{h^4} = -1$$

Now if $\mathbf{x}_n = (\frac{1}{n}, \frac{1}{n})$ and $\mathbf{y}_n = (\frac{1}{n}, \frac{1}{n^2})$, then

$$D_{12} f(\mathbf{x}_n) = 0 \neq 1 \text{ and } D_{21} f(\mathbf{y}_n) = 1 \neq -1$$

Therefore, $D_{12} f$ and $D_{21} f$ are continuous everywhere except at $(0,0)$.

(c) This has been showed in (b). ∎