Analysis II HW 2

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Problem 1:

Complete the details of the following alternative proof of Theorem 4.19. If f is not uniformly continuous, then for some $\varepsilon > 0$ there are sequences $\{p_n\}$, $\{q_n\}$ in X such that $d_X(p_n, q_n) \to 0$ but $d_Y(f(p_n), f(q_n)) \to \varepsilon$. Use Theorem 2.37 to obtain a contradiction.

Solution 1:

Theorem 4.19. Let f be a continuous mapping of a compact metric space X into a metric space Y. Then f is uniformly continuous on X.

Proof. If f is not uniformly continuous, then for some $\varepsilon > 0$ there are sequences $\{p_n\}$, $\{q_n\}$ in X such that $d_X(p_n, q_n) \to 0$ but $d_X(p_n, q_n) > \varepsilon$.

Since X is compact, then $\{p_n\}$ being an infinite subset of X, has a limit point (say p) in X (using Theorem 2.37). Similarly, $\{q_n\}$ has a limit point (say q) in X. So, there are subsequences $\{p_{n_i}\}$ of $\{p_n\}$ and $\{q_{n_i}\}$ of $\{q_n\}$, converging to p and q respectively. We have, by triangle inequality, as $n_i \to \infty$,

$$d_X(p, q) \leq d_X(p, p_{n_i}) + d_X(p_{n_i}, q_{n_i}) + d_X(q_{n_i}, q) \to 0$$

Therefore, $d_X(p, q) = 0$, i.e., $p = q$. Now since f is continuous, so $f(p_{n_i})$ and $f(q_{n_i})$ converge to $f(p) = f(q)$. So we have, by triangle inequality, as $n_i \to \infty$,

$$d_Y(f(p_{n_i}), f(q_{n_i})) \leq d_Y(f(p_{n_i}), f(p)) + d_Y(f(p), f(q_{n_i})) \to 0$$

which contradicts $d_Y(f(p_{n_i}), f(q_{n_i})) > \varepsilon$. □

Problem 2:

If E is a nonempty subset of a metric space X, define the distance from $x \in X$ to E by

$$\rho_E(x) = \inf_{z \in E} d(x, z).$$

(a) Prove that $\rho_E(x) = 0$ if and only if $x \in \overline{E}$.

(b) Prove that ρ_E is a uniformly continuous function on X, by showing that

$$|\rho_E(x) - \rho_E(y)| \leq d(x, y)$$

for all $x \in X$, $y \in X$.

Solution 2:

Proof of (a). First we shall prove that $\rho_E(x) = 0 \implies x \in \overline{E}$. Assume, to the contrary, that $\rho_E(x) = 0$ and $x \notin \overline{E}$. Then $x \in \overline{E}^C$ and since $\overline{E}$ is closed, so $\overline{E}^C$ is open. Therefore, $\exists\, r > 0$ such that

$$d(y, x) < r \implies y \in \overline{E}^C \text{ i.e., } y \notin \overline{E}$$

Thus, for every $z \in E$, $d(z, x) \geq r$ and hence

$$\rho_E(x) = \inf_{z \in E} d(z, x) \geq r > 0$$

which is a contradiction to $\rho_E(x) = 0$.

Now we shall prove that $x \in \overline{E} \implies \rho_E(x) = 0$. Suppose $x \in \overline{E}$. If $x \in E$, then

$$\rho_E(x) = d(x, x) = 0$$

If $x \notin E$, then x is a limit point of E. Then given any $\varepsilon > 0$, $\exists\ z \in E$ such that $d(z, x) < \varepsilon$. Since every $\varepsilon > 0$ is not a lower bound of $d(x, z)$, so

$$\rho_E(x) = \inf_{z \in E} d(x, z) = 0$$

□

Proof of (b). We have,

$$\rho_E(x) \leq d(x, z) \leq d(x, y) + d(y, z)$$

for any $z \in E$. Then,

$$\rho_E(x) = \inf_{z \in E} d(x, z) \leq \inf_{z \in E} (d(x, y) + d(y, z)) = d(x, y) + \rho_E(y),$$

so

$$\rho_E(x) - \rho_E(y) \leq d(x, y). \tag{1}$$

By interchanging x and y, we also get

$$\rho_E(y) - \rho_E(x) \leq d(x, y). \tag{2}$$

Since $|\rho_E(x) - \rho_E(y)|$ is either $\rho_E(x) - \rho_E(y)$ or $\rho_E(y) - \rho_E(x)$, so equations (1) and (2) give

$$|\rho_E(x) - \rho_E(y)| \leq d(x, y)$$

Now, for every $\varepsilon > 0$, take $\delta = \varepsilon$. Then

$$d(x, y) < \delta \implies d(x, y) < \varepsilon,$$

so

$$|\rho_E(x) - \rho_E(y)| < d(x, y) < \varepsilon\ \ \forall\ \ x, y \in X$$

Thus, ρ_E is uniformly continuous on X. □

Problem 3:

Suppose K and F are disjoint sets in a metric space X, K is compact, F is closed. Prove that there exists $\delta > 0$ such that $d(p, q) > \delta$ if $p \in K, q \in F$.

Show that the conclusion may fail for two disjoint closed sets if neither is compact.

Solution 3:

We shall first prove that ρ_F is a continuous positive function on K. By Problem 2(b), we have ρ_F is continuous on K. We have to show that $\rho_F(p) \neq 0$ for every $p \in K$. Assume, to the contrary, that $\rho_F(q) = 0$ for some $q \in K$. Then by Problem 2(a), we have $q \in \overline{F}$. Since F is closed, so $\overline{F} = F$ and hence $q \in F$, a contradiction since K and F are disjoint sets. Therefore, ρ_F is a continuous positive function on K. Also since K is compact, ρ_F attains its minimum value on K, i.e., $\exists\ r \in K$ such that

$$\rho_F(r) = \min_{p\in K} \rho_F(p)$$

Let $0 < \delta < \rho_F(q)$, then for any $p \in K, q \in F$, we have

$$d(p, q) \geq \rho_F(p) \geq \rho_F(r) > \delta$$

The conclusion may fail for two disjoint closed sets if neither is compact. Consider $K = \mathbb{N}$ and $F = \{n + \frac{1}{n} \mid n \in \mathbb{N}, n > 1\}$. Then for $p_n \in K$ and $q_n \in F$, we have

$$\lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} \frac{1}{n} = 0$$

i.e., $\nexists\ \delta > 0$ such that $d(p, q) > \delta$ if $p \in K, q \in F$. $\square$

Problem 4:

Let A and B be disjoint nonempty closed sets in a metric space X, and define

$$f(p) = \frac{\rho_A(p)}{\rho_A(p) + \rho_B(p)} \quad (p \in X)$$

Show that f is a continuous function on X whose range lies in $[0, 1]$, that $f(p) = 0$ precisely on A and $f(p) = 1$ precisely on B. This establishes a converse of Exercise 3: Every closed set $A \subset X$ is $Z(f)$ for some continuous real f on X. Setting

$$V = f^{-1}\left(\left[0, \frac{1}{2}\right)\right), \quad W = f^{-1}\left(\left(\frac{1}{2}, 1\right]\right),$$

show that V and W are open and disjoint, and that $A \subset V, B \subset W$. (Thus pairs of disjoint closed sets in a metric space can be covered by pairs of disjoint open sets. This property of metric spaces is called *normality*.)

Solution 4:

We shall first prove that $\rho_A(p) + \rho_B(p) \neq 0$ for every $p \in X$. Since A and B are closed, so $\overline{A} = A$ and $\overline{B} = B$. From Problem 2(a), we have $\rho_A(p) = 0$ if and only if $p \in \overline{A} = A$ and $\rho_B(p) = 0$ if and only if $p \in \overline{B} = B$. But since A and B are disjoint, it follows that $\rho_A(p) + \rho_B(p) \neq 0$. By Problem 2(b), we have $\rho_A(p)$ is continuous and $\rho_A(p) + \rho_B(p)$, being a sum of two continuous functions and $f(p)$, being a quotient of two continuous functions are continuous (using Theorem 4.9), i.e., f is a continuous function on X, with range in $[0, 1]$.

Using Problem 2(a), since A is closed, we have $\overline{A} = A$ and hence $\rho_A(p) = 0$ for $p \in A$. Since, $\rho_A(p) + \rho_B(p) \neq 0$ for every $p \in X$, we have $f(p) = 0$ on A. Similarly, $\rho_B(p) = 0$ for $p \in B$. Since $\rho_A(p) \neq 0$ when $\rho_B(p) \neq 0$ for $p \in X$, therefore $f(p) = \frac{\rho_A(p)}{\rho_A(p)} = 1$.

Since $\left[0, \frac{1}{2}\right)$ is an open set in $[0, 1]$, and since f is continuous, so $V = f^{-1}\left(\left[0, \frac{1}{2}\right)\right)$ is open. Similarly, f is continuous and $\left(\frac{1}{2}, 1\right]$ is open in $[0, 1]$, so W is open. Now we shall prove that V and W are disjoint. Suppose they are not, then there is a $x \in X$ such that $x \in V$ and $x \in W$. But then, this gives $f(x) \in \left[0, \frac{1}{2}\right)$ and $f(x) \in \left(\frac{1}{2}, 1\right]$, a contradiction. Therefore, V and W are disjoint. Also, we have proved that $p \in A \implies f(p) = 0$, i.e., $p = f^{-1}(0) \in V$. Therefore, $A \subset V$. Similarly, $B \subset W$ because $p \in B \implies f(p) = 1$. $\square$

Problem 5:

Suppose X, Y, Z are metric spaces, and Y is compact. Let f map X into Y, let g be a continuous one-to-one mapping of Y into Z, and put $h(x) = g(f(x))$ for $x \in X$.

Prove that f is uniformly continuous if h is uniformly continuous.

Prove also that f is continuous if h is continuous.

Show (by modifying Example 4.21, or by finding a different example) that the compactness of Y cannot be omitted from the hypotheses, even when X and Z are compact.

Solution 5:

Since $g : Y \to Z$ is continuous and Y is compact, so by Theorem 4.14, $g(Y)$ is compact. Also, since g is a continuous one-one mapping of a compact metric space Y onto Z, so by Theorem 4.17, $g^{-1} : g(Y) \to Y$ is continuous, and hence by Theorem 4.19, g^{-1} is uniformly continuous. Now, since $f(x) = g^{-1}(h(x))$, so f is uniformly continuous if h is uniformly continuous.

We established that $g^{-1} : g(Y) \to Y$ is continuous by Theorem 4.17. Since $f(x) = g^{-1}(h(x))$, so by Theorem 4.7, f is continuous if h is continuous.

As in Example 4.21, let $X = [0, 2\pi]$, $Y = [0, 2\pi)$ and Z be the unit circle on the plane. Define $f : X \to Y$ by

$$f(x) = \begin{cases} x, & 0 \le x < 2\pi \\ 0, & x = 2\pi \end{cases}$$

Define $g : Y \to Z$ by $g(y) = (\cos y, \sin y)$ for all $y \in [0, 2\pi)$ and $h : X \to Z$ by

$$h(x) = \begin{cases} (\cos x, \sin x), & 0 \le x < 2\pi \\ (1, 0), & x = 2\pi \end{cases}$$

Here,

$$h(x) = g(f(x)) \; \forall \; x \in X$$

Also,

$$\begin{aligned} |h(x) - h(y)| &= \sqrt{(\cos x - \cos y)^2 + (\sin x - \sin y)^2} \\ &= \sqrt{(1 - 2\cos x \cos y) + (1 - 2\sin x \sin y)} \\ &= \sqrt{2(1 - \cos(x - y))} \\ &= \sqrt{2 \cdot 2\sin^2\left(\frac{x - y}{2}\right)} \\ &= 2\left|\sin\left(\frac{x - y}{2}\right)\right| \le 2 \cdot \frac{|x - y|}{2} = |x - y| \end{aligned}$$

and since h is continuous at 2π, we have h is uniformly continuous. But f is not continous, even when X and Z are compact. □