Analysis II HW 1

Nirjhar Nath
nirjhar@cmi.aci.in

Solution 1:

Proof of (b) $\implies$ (a)

Suppose the norms $||\cdot||$ and $||\cdot||'$ induce the same topology. We denote by $B_r(x)$ and $B'_r(x)$ the open balls of radius r centered at $x \in V$ with respect to the norms $||\cdot||$ and $||\cdot||'$ respectively. Since $B_1(0)$ is open with respect to $||\cdot||$, so it is also open with respect to $||\cdot||'$. Therefore, since $0 \in B_1(0)$, so there exists $a > 0$ such that $B'_a(0) \subseteq B_1(0)$.

Now choose any $v \in V$ and $\varepsilon > 0$. Then

$$\frac{(a-\varepsilon)}{||v||'}v \in B'_a(0) \subseteq B_1(0)$$

and so,

$$\left|\left|\frac{(a-\varepsilon)}{||v||'}v\right|\right| < 1$$

i.e.,

$$(a-\varepsilon)||v|| < ||v||'$$

Since this is true for every $\varepsilon > 0$, so $a||v|| \leq ||v||'$. A symmetric argument interchanging the role of both the norms shows that there exists $b > 0$ such that $||v||' \leq b||v||$ for every $v \in V$. Therefore, for every $v \in V$, there exist $a, b > 0$ such that

$$a||v|| \leq ||v||' \leq b||v||$$

i.e., the norms $||\cdot||$ and $||\cdot||'$ are equivalent.

Proof of (c) $\implies$ (a)

Suppose (c) holds, i.e., a sequence x_n in V converges under $||\cdot|| \iff$ it converges under $||\cdot||'$ and in that case the limit under each norm is the same. Let $\alpha := \inf\{||x_n||' \mid n \in \mathbb{N}\}$ and $\beta := \sup\{||x_n||' \mid n \in \mathbb{N}\}$. Then α and β are finite since x_n converges under $||\cdot||'$. For each n, we have

$$a||x_n|| \leq ||x_n||' \leq b||x_n||$$

where $a = \dfrac{\alpha}{||x_n||}$ and $b = \dfrac{||x_n||'}{||x_n||} \leq \dfrac{\beta}{||x_n||}$. As $n \to \infty$, $a, b > 0$ and independent of $\{x_n\}$.

Solution 2:

(i) Suppose T is open in S. Then for any $t \in T$, $\exists\ r_t > 0$ such that

$$d(t,a) < r_t \text{ and } a \in S \implies a \in T$$

Define

$$V := \bigcup_{x \in X} B_{r_x}(x)$$

where $B_{r_x}(x)$ are open balls (centered at x with radius r_x) in X. We shall prove that $T = S \cap V$, as follows. We have,

$$t \in T \implies t \in S \text{ and } t \in B_{r_x}(x) \subseteq V \implies t \in S \cap V$$

Also,

$$t \in S \cap V \implies t \in S \cap B_{r_x}(x) \text{ for some } B_{r_x}(x) \implies d(t,x) < r_x \implies t \in T \text{ (as } t \in S)$$

Therefore, $T = S \cap V$.

Now we prove the other direction. Let V be an open set in X such that $T = S \cap V$. Then $t \in T \implies t \in V$. So, there exists $\varepsilon > 0$ such that the open ball $B_\varepsilon(t) \subset V$. So, $B_\varepsilon(t) \cap S \subset T$. Therefore, T is open in S.

Now we prove the next part of the problem. If T is open in S, then there exists a set V in X such that $T = S \cap V$. Since S is open in X, there exists a set U in X such that $S = X \cap U$. So we have $T = X \cap (U \cap V)$, which means that T is open in X.

Now we prove the other direction. If T is open in X, then there exists a set V in X such that $T = X \cap V$. Since S is open in X, there exists a set U in X such that $S = X \cap U$. So we have $T = S \cap (U \cap V)$, which means that T is open in S.

(ii) The analogous statement for closed sets is "A subset T of S is closed in the metric space S if and only if there exists a set F closed in X such that $T = S \cap F$. If S is closed in X, then T is closed in $S \iff T$ is closed in X."
Proof. If T is closed in S, then its complement in S, $S \setminus T$, is open in S. By definition of closed sets, there exists a set V in X such that $S \setminus T = S \cap V$. Since S is closed in X, there exists a set U in X such that $S = X \setminus U$. So we have $S \setminus T = X \setminus (U \cap V)$, which means that $T = X \setminus (U \cap V)^C$. Since the complement of an open set is closed, $(U \cap V)^C$ is closed in X, and so T is closed in X.

Now we prove the other direction. If T is closed in X, then its complement in X, $X \setminus T$, is open in X. By definition of closed sets, there exists a set V in X such that $X \setminus T = X \cap V$. Since S is closed in X, there exists a set U in X such that $S = X \setminus U$. So we have $X \setminus T = S \cap (U \cap V)$, which means that $T = S \setminus (U \cap V)^C$. Since the complement of an open set is closed, $(U \cap V)^C$ is closed in S, and so T is closed in S.

Solution 3:

(i) <u>Proof of (a) $\implies$ (b)</u>
For every $n \in \mathbb{N}$, consider the open balls $B_{\frac{1}{n}}(x)$ of radius $\frac{1}{n}$ centered at $x \in X$. Assuming (a) is true, we have at least one point in $B_{\frac{1}{n}}(x) \cap S$ for every $n \in \mathbb{N}$. Consider one point x_n from every $B_{\frac{1}{n}}(x) \cap S$.
Claim. The sequence $\{x_n\}$ converges to x.
Proof. Consider any $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon}$. Then for any $n \geq N$, we have

$$d(x_n, x) < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

i.e., x_n converges to x.
<u>Proof of (b) $\implies$ (a)</u>
Assume, to the contrary, that (b) is true but (a) is not. Then there exists $\delta > 0$ such that

$$B_\delta(x) \cap S = \phi$$

Thus, for any $a \in S, a \notin B_\delta(x)$ (i.e., $d(a, x) \geq \delta$) and any sequence $\{x_n\}$ in S, $\{x_n\}$ does not converge to x, because $d(x_n, x) \geq \delta \ \forall\ n \in \mathbb{N}$. So, no sequence in S converges to x, a contradiction.

(ii) Define

$$B_r(x) \setminus \{x\} := \{t \mid d(x,t) < r, t \neq x\}$$

and call it the deleted ball of radius r centered at x. The analogues of (a) and (b) in (i) are:

(a) The intersection of S with every deleted ball centered at x contains at least one point.

(b) There is a sequence $\{x_n\}$ in S converging to x with $x_n \neq x \; \forall \; n \in \mathbb{N}$.

Proof of (a) $\implies$ (b)
We consider $B_{\frac{1}{n}}(x) \setminus \{x\}$ instead of $B_{\frac{1}{n}}(x)$ and the remaining is same as that of (i).
Proof of (b) $\implies$ (a)
We consider $B_\delta(x) \setminus \{x\}$ instead of $B_\delta(x)$ and the remaining is same as that of (i).

(iii) Let

$$B'_r(c) = \{x \mid d(c,x) \leq r\}$$

To show that $B'_r(c)$ is closed, we need to show that its complement

$$X \setminus B'_r(c) = \{x \mid d(c,x) > r\}$$

is open. Let $y \in X \setminus B'_r(c)$. Then $d(c,y) > r$. Choose $\varepsilon > 0$ such that

$$0 < \varepsilon < d(c,x) - r$$

Then for any $z \in B_\varepsilon(y)$, by triangle inequality, we have

$$d(c,z) \geq d(c,y) - d(z,y) > d(c,y) - \varepsilon > r$$

Therefore, $z \in X \setminus B'_r(c)$. Thus, for every $y \in X \setminus B'_r(c)$, there is an open ball centered at y contained in $X \setminus B'_r(c)$, i.e., $X \setminus B'_r(c)$ is open, i.e., $B'_r(c)$ is closed.

Consider the metric space $X = \{a, b, c\}$ with the following metric:

$$d(a,b) = d(b,c) = 1, d(a,c) = 2$$

Let $r = 1$ and let $c = a$. Here, the set $\{x \mid d(c,x) \leq r\}$ is not the closure of the open ball $B_r(a)$, since it includes the point b which is not in the open ball. So, in a general metric space, the set $\{x \mid d(c,x) \leq r\}$ need not be the closure of the open ball $\{x \mid d(c,x) < r\}$.

In the Euclidean space $\mathbb{R}^n$, the closure of the set $B_r(c) = \{x \mid d(c,x) < r\}$ is the set $B'_r(c) = \{x \mid d(c,x) \leq r\}$. We denote by $\overline{A}$ the closure of set A. We want to show that $\overline{B_r(c)} = B'_r(c)$. By the first part of this problem, we have $B'_r(c)$ is the closed set containing $B_r(c)$ and $\overline{B_r(c)}$ is the intersection of all closed sets containing $B_r(c)$. So,

$$\overline{B_r(c)} \subseteq B'_r(c) \tag{1}$$

Let $x \in B'_r(c)$. If $x \in B_r(c)$, then clearly $x \in \overline{B_r(c)}$. If $x \in B'_r(c) \setminus B_r(c)$, then we claim that x is a limit point of $B_r(c)$. We prove this as follows.

Consider the sequence $\{x_n\}$ defined by

$$x_n = \frac{1}{n}c + \left(1 - \frac{1}{n}\right)x$$

Then,

$$d(x_n, c) = ||c - x_n|| = \left|\left|\left(1 - \frac{1}{n}\right)c - \left(1 - \frac{1}{n}\right)x\right|\right| = \left(1 - \frac{1}{n}\right)||c - x|| = \left(1 - \frac{1}{n}\right)r < r$$

and hence $x_n \in B_r(c)$. Also,

$$d(x_n, x) = ||x - x_n|| = \left|\left|\frac{1}{n}x - \frac{1}{n}c\right|\right| = \frac{1}{n}||x - c|| = \frac{1}{n}r > 0$$

and hence $d(x_n, x) \to 0$ as $n \to \infty$, i.e., $x_n \to x$ as $n \to \infty$ with $x_n \neq x \ \forall \ n \in \mathbb{N}$. So, x is a limit point of $B_r(c)$ and hence $x \in \overline{B_r(c)}$. Therefore,

$$B'_r(c) \subseteq \overline{B_r(c)} \tag{2}$$

So, combining equations (1) and (2), we have $\overline{B_r(c)} = B'_r(c)$.

Solution 4:

(i) <u>Proof of (a) $\implies$ (b)</u>
Suppose $f : X \to Y$ satisfies condition (a). We need to show that $f(x_n)$ converges to $f(a)$. Take any $\varepsilon > 0$. Then for any $a \in X$, there exists $\delta > 0$ such that $d_X(x, a) < \delta \implies d_Y(f(x), f(a)) < \varepsilon$. If $x_n \to a$, then there exists $N \in \mathbb{N}$ such that $d(x_n, a) < \delta \ \forall \ n \geq N$. Then $d_Y(f(x), f(a)) < \varepsilon$, i.e., $f(x)$ converges to $f(a)$ as x converges to a.
<u>Proof of (b) $\implies$ (a)</u>
Suppose $f : X \to Y$ is such that (b) holds but (a) does not hold, i.e., there exists an $\varepsilon > 0$ such that for all $\delta > 0$, there exists an x with $d_X(x, a) < \delta$ for which $d_Y(f(x), f(a)) \geq \varepsilon$. Then for each $n \in \mathbb{N}$, we can set $\delta = \frac{1}{n}$, which gives an x_n with $d_X(x_n, a) < \frac{1}{n}$ so that $d_Y(f(x_n), f(a)) \geq \varepsilon$. So, x_n converges to a. But since $d_Y(f(x), f(a)) \geq \varepsilon$, so $f(x_n)$ does not converge to $f(a)$, a contradiction to (b). Therefore, (b) $\implies$ (a).

(ii) <u>Proof of (d) $\implies$ (c')</u>
Consider any closed set C in Y. We want to prove that $f^{-1}(C)$ is closed in X. If (d) is true, then

$$f(\overline{f^{-1}(C)}) \subseteq \overline{f(f^{-1}(C))} \tag{3}$$

We also have $f(f^{-1}(S)) \subseteq S$ for any subset $S \subseteq Y$. This is true because if $x \in f(f^{-1}(S))$, then there exists $y \in f^{-1}(S)$ such that $f(y) = x$ and since $y \in f^{-1}(S)$, so $f(y) \in S$, which gives $x \in S$. Using this result in equation (3), we have

$$f(\overline{f^{-1}(C)}) \subseteq \overline{C} = C$$

because C is closed in Y. We also have $T \subseteq f^{-1}(f(T))$ for any subset $T \subseteq X$. This is true because if $x \in T$ and $x \notin f^{-1}(f(T))$, it gives $f(x) \notin f(T)$, a contradiction since $x \in T$. Therefore, using this result, we have

$$\overline{f^{-1}(C)} \subseteq f^{-1}(f(\overline{f^{-1}(C)})) \subseteq f^{-1}(C)$$

Also since $f^{-1}(C) \subseteq \overline{f^{-1}(C)}$, so $f^{-1}(C) = \overline{f^{-1}(C)}$, i.e., $f^{-1}(C)$ is closed.
Proof of (c') $\implies$ (d)
Consider any subset $S \subseteq X$. We need to prove that $f(\overline{S}) \subseteq \overline{f(S)}$ assuming (c') is true. Since $\overline{f(S)}$ is closed in Y, therefore by (c'), $f^{-1}(\overline{f(S)})$ is closed in X. Using a result of the previous proof, we get

$$S \subseteq f^{-1}(f(S)) \subseteq f^{-1}(\overline{f(S)}) \tag{4}$$

Since $f^{-1}(\overline{f(S)})$ is closed in X, so $f^{-1}(\overline{f(S)}) = \overline{f^{-1}(\overline{f(S)})}$. Therefore, equation (4) becomes

$$S \subseteq f^{-1}(\overline{f(S)}) \implies f(S) \subseteq \overline{f(S)}$$

Again, using a result of the previous proof, we have

$$f(\overline{S}) \subseteq f(f^{-1}(\overline{f(S)})) \subseteq \overline{f(S)}$$

Proof that (c) and (c') are equivalent
We observe that $f^{-1}(S^C) = (f^{-1}(S))^C$ for any subset $S \subseteq Y$. This is because

$$x \in f^{-1}(S^C) \iff f(x) \in S^C \iff f(x) \notin S \iff x \notin f^{-1}(S) \iff x \in (f^{-1}(S))^C$$

If A is closed in Y, then A^C is open in Y, then $f^{-1}(A^C)$ is open in X and therefore by the previous observation, $(f^{-1}(A))^C$ is open in X. Therefore, $f^{-1}(A)$ is closed in X and hence (c) $\implies$ (c'). Similarly, (c') $\implies$ (c), and we are done.

(iii) (d') For any subset T of Y, $f^{-1}(T^\circ) \subseteq (f^{-1}(T))^\circ$, where A° denotes the interior of A.
Proof of (d) $\implies$ (d')
Suppose (d) holds, i.e., for any subset $S \subseteq X$, $f(\overline{S}) \subseteq \overline{f(S)}$. Let T be a subset of Y, and let U be an open set containing $f^{-1}(T^\circ)$. Then $f(U)$ is an open set containing T°, which means

$$f(U) \subseteq T \implies f^{-1}(f(U)) \subseteq f^{-1}(T) \implies U \subseteq f^{-1}(T)$$

which means that $f^{-1}(T^\circ) \subseteq (f^{-1}(T))^\circ$.
Proof of (d') $\implies$ (d)
Suppose (d') holds, i.e., for any subset $T \subseteq Y$, $f^{-1}(T^\circ) = (f^{-1}(T))^\circ$. Let S be a subset of X, and let U be an open set containing $\overline{S}$. Then $f(U)$ is an open set containing $f(\overline{S})$, which means

$$f(U) \subseteq \overline{f(S)} \implies f^{-1}f(U) \subseteq f^{-1}(\overline{f(S)}) \implies U \subseteq f^{-1}(\overline{f(S)}) \implies \overline{S} \subseteq f^{-1}(\overline{f(S)})$$

which gives $f(\overline{S}) \subseteq \overline{f(S)}$.

Solution 5:

(i) We consider the max metric in $X_1 \times X_2$ given by

$$d((x_1, y_1), (x_2, y_2)) = \max(d((x_1, x_2), (y_1, y_2))$$

To show that open sets in ($X_1 \times X_2$, max metric) are precisely unions of sets of the form (open ball of X_1 $\times$ open ball of X_2), we need to prove two things:

(a) Every open set in $X_1 \times X_2$ can be expressed as a union of sets of the form (open ball of X_1 × open ball of X_2).

(b) Every union of sets of the form (open ball of X_1 × open ball of X_2) is an open set in $X_1 \times X_2$.

Proof of 1.
Let $B_r^{X_1 \times X_2}((x_0, y_0))$ be an open ball in $X_1 \times X_2$, of radius r centered at (x_0, y_0). We have,

$$\begin{aligned} B_r^{X_1 \times X_2}((x_0, y_0)) &= \{(x, y) \mid \max(d_{X_1}(x, x_0), d_{X_2}(y, y_0)\} \\ &= \{(x, y) \mid d_{X_1}(x, x_0) < r, d_{X_2}(y, y_0) < r\} \\ &= B_r^{X_1}(x_0) \times B_r^{X_2}(y_0) \end{aligned}$$

Therefore, every open set in $X_1 \times X_2$ can be expressed as a union of sets of the form (open ball of X_1 × open ball of X_2).
Proof of 2.
Consider a set of the form

$$B_r^{X_1}(x_0) \times B_s^{X_2}(y_0) = \{(x, y) \mid d_{X_1}(x, x_0) < r, d_{X_2}(y, y_0) < s\}$$

We want to prove that this set is open in $X_1 \times X_2$. Choose t such that

$$0 < t < \min(r - d_{X_1}(x, x_0), s - d_{X_2}(y, y_0))$$

Then

$$B_t^{X_1 \times X_2}((x, y)) = \{(x', y') \mid d_{X_1}(x, x') < t, d_{X_2}(y, y') < t\}$$

and by triangle inequality, for $(a, b) \in B_t^{X_1 \times X_2}((x, y))$, we have

$$d_{X_1}(x_0, a) \leq d_{X_1}(x_0, x) + d_{X_1}(x, a) < d_{X_1}(x_0, x) + t < r$$

and,

$$d_{Y_1}(y_0, b) \leq d_{X_2}(y_0, y) + d_{X_2}(y, b) < d_{X_2}(y_0, y) + t < s$$

and hence,

$$(a, b) \in B_r^{X_1}(x_0) \times B_s^{X_2}(y_0)$$

It follows that

$$B_t^{X_1 \times X_2}((x, y)) \subseteq B_r^{X_1}(x_0) \times B_s^{X_2}(y_0)$$

i.e., the open ball $B_t^{X_1 \times X_2}((x, y))$ is contained in $B_r^{X_1}(x_0) \times B_s^{X_2}(y_0)$. Therefore, every union of sets of the form (open ball of X_1 × open ball of X_2) is an open set in $X_1 \times X_2$.

Yes, we can replace "open ball" with "arbitrary open set" of the respective X_i spaces. This is because the definition of an open set in a metric space is a set that contains an open ball around every point in the set. Hence, every open set in a metric space can be expressed as a union of open balls.

The two projection maps $\pi_1 : X_1 \times X_2 \to X_1$ and $\pi_2 : X_1 \times X_2 \to X_2$ are continuous.

A map $f : X \to Y$ between two metric spaces (X, d_X) and (Y, d_Y) is continuous if for every open set U in Y, the preimage $f^{-1}(U)$ is an open set in X. In this case, for each open set U in X_1 (or X_2), the preimage $\pi_1^{-1}(U) = U \times X_2$ (or $\pi_2^{-1}(U) = X_1 \times U$) is an open set in $X_1 \times X_2$, because it is a product of open sets. Hence, both π_1 and π_2 are continuous maps.

(ii) Let (X, d) be a metric space and p be a fixed point in X. First, observe that the function $x \to d(p, x)$ is continuous, since for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$d(p, x) < \delta \implies |d(p, x) - d(p, y)| < \epsilon$$

Let $(x, y) \in X \times X$ and $\epsilon > 0$. Let $\delta_1 > 0$ and $\delta_2 > 0$ be such that

$$d(p, x) < \delta_1 \implies |d(p, x) - d(p, z)| < \frac{\epsilon}{2} \text{ and } d(p, y) < \delta_2 \implies |d(p, y) - d(p, z)| < \frac{\epsilon}{2}$$

Define $\delta = \min(\delta_1, \delta_2)$. Let $(u, v) \in X \times X$ be such that $d(x, u) < \delta$ and $d(y, v) < \delta$. Then we have $d(p, u) < \delta_1$ and $d(p, v) < \delta_2$, so

$$\begin{aligned} |d(x, y) - d(u, v)| &= |d(p, x) - d(p, u) + d(p, y) - d(p, v)| \\ &\leq |d(p, x) - d(p, u)| + |d(p, y) - d(p, v)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Hence, $(x, y) \to d(x, y)$ is continuous at (x, y). Since this holds for every $(x, y) \in X \times X$, it follows that the function $(x, y) \to d(x, y)$ is continuous on $X \times X$.