Analysis 2 Final

- Explain everything. You may use any standard result, but refer to it precisely and justify why it is applicable. Exception: for material from one variable calculus you may explain briefly.
- **Write your answers on loose sheets. Label each *side* you use (not just sheets) as follows.** At the top of each side of each page (i) write on the left your name and roll number (ii) write on the right: Problem i page x of y.
- **Upload all answers on moodle before handing in the exam.** It is your uploaded answers that will be graded (and they will be needed for the next item too), so scan and upload everything carefully.
- An optional Quiz will ask you to correct your own final (with comments) and fix all mistakes. Details will be sent by email. Deadline: May 10.

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1. Let $f(x,y) = x^2(y-2) - y^2$.

   (i) Find critical points and local extrema of f on $\mathbb{R}^2$ and classify them. Clearly show the reasons for your deductions including any hypotheses that need to be satisfied for the procedure. Are there global extrema?

   (ii) Why is f guaranteed to have global extrema when restricted to the set $\{(x,y) | x^2+y^2 \le 1\}$? Find these global extrema.

   (iii) Write $f(5+h, 3+k) =$ (quadratic Taylor polynomial in terms of variables h, k) $+ r(h,k)$. Specify the order of the error term $r(h,k)$ by stating the value of an appropriate limit involving $r(h,k)$. You should write your polynomial in terms of product(s) of matrices/vectors as appropriate and your answer should involve no letters other than h and k. You need not carry out the matrix multiplication(s).

2. Let $f(x,y) = y^2 - 3x^2 - x^3$.

   (i) Find all points (x,y) at which the inverse function theorem applies to the map $g(x,y) = (x, f(x,y))$ from $\mathbb{R}^2$ to itself. Find the derivative (matrix in standard basis) of the local inverse function of g at the point $(1,0) = g(1,2)$.

   (ii) Find a basis of the tangent space at (1,2) to the level set $M = \{(x,y) | f(x,y) = 0\} \subset \mathbb{R}^2$. Find the equation(s) describing the geometric tangent space.

   (iii) Find all points on M around which one can locally solve for one coordinate (of points on M) uniquely in terms of the other. Explain.

3. Consider the function $f(x,y,z) = |x|^3 + |y|^3 + |z|^3$ from $\mathbb{R}^3$ to $\mathbb{R}$.

   (i) Find the largest integer k such that f is a C^k function. (C^0 stands for continuous functions.)

   (ii) Let $h(x,y,z)$ be a continuous function from $\mathbb{R}^3$ to $\mathbb{R}^2$. Subject to the constraint $h(x,y,z) = (0,0)$, is f guaranteed to have a global maximum? A global minimum? (Assume $h^{-1}(0,0) \neq \phi$)

   (iii) Subject to the constraints $x^6 - z = 0$ and $y^3 - z = 0$, see that the given function f has an obvious extremum and identify it. Does the Lagrange multiplier theorem apply to detect this extremum? What does Lagrange multiplier method give if the objective function is $g(x,y,z) = y$ subject to the same constraints? (Apply the method directly using the given formulas – do not reformulate the problem in terms of some other equation(s). You do not need complicated calculations to answer this question.)

4. The symplectic group G is defined below, where J is a $2n \times 2n$ matrix written in terms of four $n \times n$ blocks and $I = n \times n$ identity matrix. Observe that $J^t = -J$ and $J^2 = -$ identity matrix ($2n \times 2n$).

$$G = \{2n \times 2n \text{ matrix } A \,|\, A^t J A = J\}, \qquad J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

Show that G is a smooth manifold and find its dimension. You may use a standard method to produce manifolds as a black box. In order to do so you should (i) state the method precisely (ii) name without proof what major result(s) it is based on, and (iii) sketch *very briefly* how the method produces charts and transition functions.

5. (Reserve question – attempt only if you could not do something above.) Carefully state and prove the mean value theorem for a function f from a closed interval $[a, b]$ to $\mathbb{R}^k$. This is straight from the text, so I am looking for a perfect exposition.

Short questions

A. True or false with justification. Given: $p \in$ open U in some Euclidean space and a function $f : U \to \mathbb{R}^n$.

(i) Suppose f is differentiable at p and γ is the "straight line" curve based at p, namely $\gamma(t) = p + tv$, where v is any vector. Then $f \circ \gamma$ is differentiable at 0 and $(f \circ \gamma)'(0) = f'(p)(v)$.

(ii) Suppose f has all directional derivatives at p. Then f is continuous at p.

(iii) Suppose f has all partial derivatives at and near p and they are continuous at p. Then f is continuous at p.

(iv) Suppose f has all directional derivatives at and near p and they are continuous at p. Then the total derivative of f exists at p and is continuous at p.

B. Let f be a continuous function from an open ball $B = B(a, r)$ in $\mathbb{R}^2$ to $\mathbb{R}$. Is $f(B)$ necessarily open? Is $f(B)$ necessarily bounded? If f is from all of $\mathbb{R}^2$ to $\mathbb{R}$, does the answer to either question (still about $f(B)$) change?

C. Identify $(x, y) \in \mathbb{R}^2$ with the complex number $x + iy$ and define the operator $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$. Calculate $\frac{\partial f}{\partial \bar{z}}$ for a holomorphic function f.

D. We know that if the derivative of a C^1 function $f : U \to V$ between open sets of $\mathbb{R}^n$ is invertible at a point, then f has a differentiable local inverse g near that point. Show that for any positive integer k, if f is C^k, then so is g. Assume the result that composition of C^k functions is C^k. If you cannot do what is asked, you may prove this latter result instead for some credit.