Midsem problem 4 solution and grading scheme

March 2023

1 Part (i)

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^{n'}$ be continuous functions such that f is differentiable at p and g is differentiable at $\tilde{p} = f(p)$. Then the composite $g \circ f : \mathbb{R}^n \to \mathbb{R}^{n'}$ is differentiable at p with its derivative given by $g \circ f(p) = g'(f(p))f'(p)$ Marks for this part: 2.

2 Part (ii)

Showing $\frac{1}{g}$ is differentiable is a straightforward application of the chain rule. $x \mapsto \frac{1}{x}$ is a differentiable function away from 0. Since g is never zero, from part (i), we are done. An alternate approach would be to use first principles. Marks for this: 1.
One mistake which a few had done was to use the product rule on $g.\frac{1}{g}$. This cannot be done because it needs to be shown that $\frac{1}{g}$ is differentiable.

To show that $fg(x)$ is differentiable, one could apply the chain rule to the composite $\mathbb{R}^n \to \mathbb{R}^2 \to \mathbb{R}$,

$$x \mapsto (f(x), g(x)) \mapsto f(x)g(x)$$

Since f and g are differentiable, the first map is differentiable and has derivative $\begin{bmatrix} f'(x) \\ g'(x) \end{bmatrix}$. The second map, $(x, y) \mapsto (x, y)$ is a polynomial function and hence is differentiable and has derivative $\begin{bmatrix} y & x \end{bmatrix}$. Hence by applying chain rule, the derivative of the composite at (x, y) is $\begin{bmatrix} g(x) & f(x) \end{bmatrix} \begin{bmatrix} f'(x) \\ g'(x) \end{bmatrix} = g(x)f'(x) + f(x)g'(x)$ Marks distribution for this method- 1.5 for showing the correct setup of the composite function. 1 to show that xy is differentiable and to mention its derivative. 1 to state that $x \mapsto (f(x), g(x))$ is differentiable. 1.5 for correct use of chain rule.

Another method is use to use first principles

$$\begin{aligned}\frac{|fg(x+h) - fg(x) - f'(x)g(x)h - f(x)g'(x)h|}{|h|} &= \frac{1}{|h|}|f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) \\ &- f(x)g(x) - f'(x)g(x)h - f(x)g'(x)h| \\ &\leq \frac{|f(x+h)g(x+h) - f(x+h)g(x) - f(x+h)g'(x)h|}{|h|} \\ &+ \frac{|f(x+h)g(x) - f(x)g(x) - f'(x)g(x)h|}{|h|} \\ &+ \frac{|(f(x+h) - f(x))g'(x)h|}{|h|}\end{aligned}$$

Fixing x, the first term tends to zero as $h \to 0$ because of differentiability of f and $f(x+h)$ tends to $f(x)$ by continuity of f. The second term also tends to zero as h tends to 0 by differentiability of g. The third term also goes to zero because of the property of the operator norm, that is $|g'(x)h| \leq ||g'||_{op}|h|$, so $\frac{|(f(x+h)-f(x))g'(x)h}{h}| \leq |f(x+h) - f(x)|||g'||_{op} \to 0$ by continuity. Hence fg is differentiable at x. Since x was arbitrary, fg is differentiable.

Marks distribution for this method- 1 for the correct use of triangular inequality, 1 each for showing the first two terms tend to zero and 2 for showing the last term tends to 0(0.5 for mentioning operator norm property).

A common mistake was to assume that the partial derivatives were continuous, hence used that to show partial derivatives of fg were continuous, implying fg was C^1. This cannot be assumed because the product rule works even for differentiable functions which are not C^1. Another mistake was using the existence of partial derivatives to imply differentiability. This cannot be done, as evidenced by many examples one may have seen.