1 ANA-2 MIDSEM QN-3

A function $f : (X, d_X) \to (Y, d_Y)$ between between metric spaces is called $uniformly\ continuous$ if for every pair of points x_1, x_2 in X the following is true: $\cdots$ (Complete the sentence). Using problem 2 or otherwise show that if f is continuous and X is compact, then (i) f(X) is compact (ii) f is uniformly continuous.

Solution :-

[**1 mark**] A function $f : (X, d_X) \to (Y, d_Y)$ between between metric spaces is called $uniformly\ continuous$ if **for all** $\epsilon > 0$ **there exists** $\delta > 0$ **such that for any** $x_1, x_2 \in X$ **we have** $d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2)) < \epsilon$

(i) [**3 marks**] We want to show that $f(X)$ is compact, Let $\{O_\alpha\}_{\alpha\in A}$ is an open cover of $f(X)$ then we notice that $\{f^{-1}(O_\alpha)\}_{\alpha\in A}$ is an open cover of X because each $f^{-1}(O_\alpha)$ is open because O_α is open and

$$x \in \bigcup_{\alpha\in A} f^{-1}(O_\alpha) \Longleftrightarrow \exists\alpha \in A\ (x \in f^{-1}(O_\alpha)) \Longleftrightarrow \exists\alpha \in A\ (f(x) \in O_\alpha)$$

$$\Longleftrightarrow f(x) \in f(X) \Longleftrightarrow x \in X$$

Where the second last equivalence is true because $\{O_\alpha\}_{\alpha\in A}$ is an open cover of $f(X)$. Now since $\{f^{-1}(O_\alpha)\}_{\alpha\in A}$ is an open cover of X it must have a finite subcover, hence say $\{f^{-1}(O_{\alpha_i})\}_{i=1}^n$ is the finite subcover of $\{f^{-1}(O_\alpha)\}_{\alpha\in A}$, then we see that $\{O_{\alpha_i}\}_{i=1}^n$ is an open cover of $f(X)$ because

$$X = \bigcup_{i=1}^n f^{-1}(O_{\alpha_i}) \implies f(X) = f(\bigcup_{i=1}^n f^{-1}(O_{\alpha_i})) = \bigcup_{i=1}^n f(f^{-1}(O_{\alpha_i})) \subseteq \bigcup_{i=1}^n O_{\alpha_i}$$

Hence we have shown that $\{O_{\alpha_i}\}_{i=1}^n$ is an open cover of $f(X)$ which is also a finite subcover of $\{O_\alpha\}_{\alpha\in A}$, hence proved $f(X)$ is compact.
Note :- it is not sufficient to show that $f(X)$ has some open finite cover, $\{Y\}$ is a finite open cover of $f(X)$.

(ii) [**4 marks**] We want to show that f is uniformly continuous, Suppose that f is not uniformly continuous then $\exists\ \epsilon > 0$ such that $\forall\ \delta > 0\ \ \exists x, y \in X$ such that $d_X(x, y) < \delta$ but $d_Y(f(x), f(y)) \geq \epsilon$, this means that $\forall\ n \in \mathbb{N}$ $\exists x_n, y_n$ such that $d_X(x_n, y_n) < \dfrac{1}{n}$ but $d_Y(f(x_n), f(y_n)) \geq \epsilon$ Now since X is compact and $\{x_n\}$ is a sequence, using problem 2 part (iii), we extract a convergent sub-sequence of $\{x_n\}$ say $\{x_{n_k}\}$ and $x_{n_k} \to x$, then we see that

$y_{n_k} \to x$ also because if $N_1 \in \mathbf{N}$ such that $k \geq N_1 \implies d_X(x_{n_k}, y_{n_k}) < \dfrac{\epsilon'}{2}$ and if $k \geq N_2 \implies d_X(x_{n_k}, x) < \dfrac{\epsilon'}{2}$ then by triangle inequality we have $d_X(y_{n_k}, x) \leq d_X(x_{n_k}, y_{n_k}) + d_X(x_{n_k}, x) < \epsilon'$ for all $k > \max(N_1, N_2)$.
Now we see since $x_{n_k}, y_{n_k} \to x$ we have $f(x_{n_k}), f(y_{n_k}) \to f(x)$ by continuity. hence there exists $K_1, K_2 \in \mathbf{N}$ such that
$k \geq K_1 \implies d_Y(f(x_{n_k}), f(x)) < \dfrac{\epsilon}{2}$ and
$k \geq K_2 \implies d_Y(f(y_{n_k}), f(y)) < \dfrac{\epsilon}{2}$ hence
$d(f(x_{n_k}), f(y_{n_k})) < \epsilon$ for all $k > \max(K_1, K_2)$ which is a contradiction, hence proved.