Midsem Question 2

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(i) A point $x \in K$ is said to be a limit point of A if the intersection of A with every open neighborhood of x contains at least one point other than x. 0.5 marks for a fully correct statement. Other equivalent formulations in terms of neighbourhoods/balls/open sets have been given marks.

We will first prove that a finite set in a metric space can't have any limit points. Let A be a finite set with limit point a. Consider $d = \min_{x \in A \setminus \{a\}} \{d(x,a)\}$. d is well defined as A is finite and > 0. Now, $B_d(a) \cap A \setminus \{a\} = \phi$ else it will contradict that $d = \min_{x \in A \setminus \{a\}} \{d(x,a)\}$. This proves A cannot have any limit points. Here, A is finite and hence, it has 0 limit points. 0.5 marks for the correct value and 0.5 marks for the proof. Some have missed the part that a can be in A and hence d becomes 0 if they take minimum of d(x,a) for all x in A. Marks have been deducted for that.

Adherent points of A = limit points of $A \cup$ points of A and as A has no limit points, number of adherent points = 2023. 0.5 marks for the correct value and reasoning.

(ii) Suppose there is no limit point. Then each $x \in K$ has an open neighborhood U_x such that $U_x \cap A \setminus \{a\} = \phi$. Then $\{U_x\}_{x \in K}$ is an open cover of K and as K is compact, it has a finite subcover and hence, we get a finite subcover of A but $|U_x \cap A| \leq 1 \,\forall\, x$ which implies A is finite. 1 marks have been given for correctly forming the open cover and extracting a finite subcover. 1 marks have been given for completing the proof and correctly listing other details.

(iii) If $\{x_n\}$ has only finitely many distinct terms then by PHP, $\exists x \in K$ and a sequence $\{n_k\}$ with $n_1 < n_2 < \ldots$, such that

$$x_{n_1} = x_{n_2} = \cdots = x$$

The subsequence $\{x_{n_k}\}$ is a constant sequence and hence, converges to x.
Else let A be the range of $\{x_n\}$. A is an infinite subset of a compact set K and hence, has a limit point in K(say x) by previous part. Now, we choose n_1 such that $d(x_{n_1}, x) < 1$ and continue in this manner such that having chosen $n_1, n_2, \ldots, n_{k-1}$, we choose a natural number $n_k > n_{k-1}$ such that $d(x_{n_k}, x) < \frac{1}{k}$. This can be done as $B_r(x) \cap A \setminus \{x, x_1, \ldots, x_{n_{k-1}}\}$ is always infinite for all $r > 0$. This obtained subsequence x_{n_k} converges to $x \in K$. 0.5 have been given for handling the case when A is finite. 1 marks have been given for correctly constructing a convergent subsequence. A lot of students have not maintained the property that indices of a subsequence is increasing for which marks have been deducted. 0.5 marks have been given for other details and completing the proof.

(iv) Let $A = \{\frac{1}{n}, n \in \mathbb{N}\}$ and $K = (0, 2]$ then clearly A is an infinite subset of K and K is bounded. The sequence $\frac{1}{n}$ converges to 0 in $\mathbb{R}$ and hence, A can only have limit point 0 but 0 is not in K and hence, it has no limit point in K. 0.5 marks have been given for giving a correct example and 0.5 marks for justification.

Yes, it is necessary for A to have a limit point in $\mathbb{R}^n$ as we can take closure of K which is then closed and bounded and hence, by heine borel, it is compact in $\mathbb{R}^n$. By second part, A must have a limit point in $\text{cl}(K) \subset \mathbb{R}^n$ and hence, a limit point in $\mathbb{R}^n$. 1 marks have been given for stating Yes along with proper justification and use of heine borel.