Analysis II Midsem

March 11, 2023

Problem 1(i) : Let $p \in U \subset \mathbb{R}^n$ and let $f : U \longrightarrow \mathbb{R}^k$ be a function. Define $f'(p)$. Include any additional hypothesis needed on the nature of U and/or f. Give a parallel definition of $f''(p)$

Solution : We need U to be open in $\mathbb{R}^n$. $f'(p)$ is defined as the unique linear map (if exists) $T : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ for which $\lim\limits_{h \longrightarrow 0} \dfrac{f(p+h) - f(p) - T(h)}{||h||} = 0$.

For the next part, in addition to U being open, we also need f should be differentiable in a neighbourhood $N \subset U$ of p. $f' : N \longrightarrow L(\mathbb{R}^n, \mathbb{R}^k)$ is a function. $f''(p)$ is defined as the unique linear map (if exists), $S : \mathbb{R}^n \longrightarrow L(\mathbb{R}^n, \mathbb{R}^k)$ st. $\lim\limits_{h \longrightarrow 0} \dfrac{f'(p+h) - f'(p) - S(h)}{||h||} = 0$

Problem 1(ii) : Now let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $f(x, y) = (xy^2, x^2 + y^4)$ and let $p = (1, 1)$. Justify why $f'(p)$ exists and specify $f'(p)$ in terms of finitely many real nos, stating the precise way in which these numbers describe $f'(p)$ as defined in part (i). Specify all points at which f is $\mathcal{C}^1$

Solution : We'll use that f is $\mathcal{C}^1$ on open $U \subseteq \mathbb{R}^2 \iff D_1 f, D_2 f$ exists and are continuous on U

Say, $f = (f_1, f_2)$, $f_1(x, y) = xy^2$, $f_2(x, y) = x^2 + y^4$, then f_1, f_2 are both polynomials, so are $\dfrac{\partial f_1}{\partial x} = y^2$, $\dfrac{\partial f_1}{\partial y} = 2xy$, $\dfrac{\partial f_2}{\partial x} = 2x$ and $\dfrac{\partial f_2}{\partial y} = 4y^3$, so they are continuous. Hence, by slotwise continuity of component functions, $D_1 f = \left(\dfrac{\partial f_1}{\partial x}, \dfrac{\partial f_2}{\partial x}\right)$, $D_2 f = \left(\dfrac{\partial f_1}{\partial y}, \dfrac{\partial f_2}{\partial y}\right)$ are continuous on $\mathbb{R}^2$. So, f in $\mathcal{C}^1$ on $\mathbb{R}^2$ (this also proves $f'(p)$ exists).

Now, $D_1 f(p) = \left(\dfrac{\partial f_1}{\partial x}\Big|_p, \dfrac{\partial f_2}{\partial x}\Big|_p\right) = (1, 2)$, $D_2 f(p) = \left(\dfrac{\partial f_1}{\partial y}\Big|_p, \dfrac{\partial f_2}{\partial y}\Big|_p\right) = (2, 4)$

These 4 real nos. $1, 2, 2, 4$ form the matrix of $f'(p)$. $f'(p)$ is given by the linear map $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ 2x + 4y \end{pmatrix} \blacksquare$$

Problem 1(iii) : Specify all points at which f is $\mathcal{C}^2$. For $p = (1,1)$ specify $f''(p)$ in terms of finitely many real nos, stating the precise way in which these numbers describe $f''(p)$ as defined in part (i). What is the minimum no. of distinct calculations you need to do to describe $f''(p)$?

Solution : Note that $f''(p) : \mathbb{R}^2 \longrightarrow L(\mathbb{R}^2, \mathbb{R}^2))$ and $L(\mathbb{R}^2, \mathbb{R}^2) \cong \mathbb{R}^4$, so $f''(p)$ can be thought as a linear map $\mathbb{R}^2 \longrightarrow \mathbb{R}^4$ and can be described by a 4×2 matrix (if exists). Now, to find the points where f is $\mathcal{C}^2$, we again use that f is $\mathcal{C}^2 \iff f'$ is $\mathcal{C}^1 \iff$ partial derivatives of f' exist and are continuous on $\mathbb{R}^2$

Now, $f' : \mathbb{R}^2 \longrightarrow L(\mathbb{R}^2, \mathbb{R}^2)$ can be thought as a map $\mathbb{R}^2 \longrightarrow \mathbb{R}^4$. At each $(a,b) \in \mathbb{R}^2$, $f'(a)$ is given by the linear map whose matrix is $\begin{pmatrix} b^2 & 2ab \\ 2a & 4b^3 \end{pmatrix}$, which corresponds to the vector $(b^2, 2ab, 2a, 4b^3)$.

Thus $f' : \mathbb{R}^2 \longrightarrow \mathbb{R}^4$, $(a,b) \longmapsto (b^2, 2ab, 2a, 4b^3)$. Again as before, component functions are polynomials, so partial derivatives of f' are again polynomials, so continuous everywhere on $\mathbb{R}^2$. This shows that f' is $\mathcal{C}^1$, I.e. f is $\mathcal{C}^2$ on $\mathbb{R}^2$.

Now, $f''(p)$ can be described by the corresponding hessian matrix -

$$\begin{pmatrix} \left.\frac{\partial^2 f_1}{\partial x^2}\right|_p & \left.\frac{\partial^2 f_1}{\partial y \partial x}\right|_p \\ \left.\frac{\partial^2 f_1}{\partial x \partial y}\right|_p & \left.\frac{\partial^2 f_1}{\partial y^2}\right|_p \\ \left.\frac{\partial^2 f_2}{\partial x^2}\right|_p & \left.\frac{\partial^2 f_2}{\partial y \partial x}\right|_p \\ \left.\frac{\partial^2 f_2}{\partial x \partial y}\right|_p & \left.\frac{\partial^2 f_2}{\partial y^2}\right|_p \end{pmatrix} = \left.\begin{pmatrix} 0 & 2y \\ 2y & 2x \\ 2 & 0 \\ 0 & 12y^2 \end{pmatrix}\right|_p = \begin{pmatrix} 0 & 2 \\ 2 & 2 \\ 2 & 0 \\ 0 & 12 \end{pmatrix}$$

These nos. $0, 2, 12$ describe $f''(p)$. $f'(a,b)$ is the linear map $\big((x,y) \longmapsto (b^2x + 2aby, 2ax + 4b^3y)\big)$

We know, if $g'(p)$ exists then, $g'(p)(v) = \lim_{t \longrightarrow 0} \frac{g(p+tv) - g(p)}{t}$ (put $h = tv$ in limit definition of $g'(p)$ and let $t \longrightarrow 0$) Here we know that $f''(p)$ exists.

So, $f''(p)(a,b) = \lim_{t \longrightarrow 0} \frac{f'(1+ta, 1+tb) - f'(1,1)}{t}$. Now, $f''(p)$ associates each point $(a,b) \in \mathbb{R}^2$ to a linear map $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$. So, $f''(p)(a,b)$ is a linear map, which sends the point $(x,y) \in \mathbb{R}^2$ to

$$f''(p)(a,b)(x,y) = \lim_{t \longrightarrow 0} \frac{f'(1+ta, 1+tb)(x,y) - f'(1,1)(x,y)}{t}$$

$$= \lim_{t \longrightarrow 0} \frac{\big((1+tb)^2x + 2(1+ta)(1+tb)y, 2(1+ta)x + 4(1+tb)^3y\big) - \big(x+2y, 2x+4y\big)}{t}$$

$$= \lim_{t \longrightarrow 0} \left(\Big(\frac{(1+tb)^2 - 1}{t}\Big) \cdot x + \Big(\frac{(1+ta)(1+tb) - 1}{t}\Big) \cdot 2y \ , \ \Big(\frac{(1+ta) - 1}{t}\Big) \cdot 2x + \Big(\frac{(1+tb)^3 - 1}{t}\Big) \cdot 4y \right)$$

$$= (2bx + 2(a+b)y \ , \ 2ax + 12by)$$

Now, to describe $f''(p)$, we need to compute all 8 entries of the hessian matrix, but as f_1, f_2 are smooth, by equality of mixed partial derivatives, $(f_1)_{xy} = (f_1)_{yx}$, $(f_2)_{xy} = (f_2)_{yx}$, so we don't need to re-calculate same value twice. So, we need to do 6 distinct calculations ■

Grading Scheme : (Total 8 marks)

(i) (Total 2 marks) 1 marks for correctly defining $f'(p)$ (0.5 marks, depending on what information you've missed)

1 marks for correctly defining $f''(p)$, Note that if you give same definition what you used for $f'(p)$, by replacing f with f' in the limit, you won't get any marks. You need to specify things like f' is a map $U \longrightarrow L(\mathbb{R}^n, \mathbb{R}^k)$, $f''(p)$ is a linear map, you must mention its domain and codomain. (0.5 marks, depending on what other information you've missed)

(ii) (Total 3 marks) (0.5 + 0.5) = 1 marks for justifying why $f'(p)$ exists and why f is $\mathcal{C}^1$

1 marks for calculating matrix of f'(p)/ partial derivatives correctly

1 marks for describing the linear map f'(p) in terms of some finitely many reals, Note, only calculating jacobian matrix will not give you marks, you need to atleast describe the linear map in some way. I.e. write - these real nos form the entries of this matrix and $f'(p)$ is given by the map $v \longmapsto Jv$ (0.5 marks partial credit, depending on how you describe)

(iii) (Total 3 marks)

1 marks for calculating the hessian matrix / second order partial derivatives correctly

1 marks for minimum no. of distinct calculations (0.5 for correct answer and 0.5 for justification)

1 marks for finding where f is $\mathcal{C}^2$ and describing $f''(p)$ like as in part (ii) . This description is lengthy, any sketch of the description of $f''(p)(x, y)(a, b)$ or describing how the hessian matrix gives the map $f''(p)$ will give you full marks, but you have to describe something. Only calculating hessian matrix will not give you any marks.

Some Comments :

- For defining $f''(p)$, some of you have used norm in the numerator $\|f'(p+h) - f'(p) - T(h)\|$. Note that, for $f'(p)$, the norm in the numerator is your standard norm in $\mathbb{R}^k$, but for $f''(p)$, this norm is Operator norm. You have to mention this.
  In these definitions, some of you have not mentioned that U should be open to define $f'(p)$. Derivative is a "local property", to have a notion of "local-ness", you domain must contain vectors sufficiently close to p.
- Some of you have misunderstood $f'(p)$ as a vector. $f'(p) = \lim_{h \longrightarrow 0} \frac{f(p+h) - f(p)}{\|h\|}$. These are partial derivatives, not your total derivative. Similar mistake for $f''(p)$. Some of you have written $f''(p)$ is a 2×2 matrix or a vector $(0, 2, 2, 12)$. This is also wrong. $f''(p)$ should be interpreted as a 4×2 matrix. Some of you have written $f'(p) : \mathbb{R}^2 \longrightarrow L(\mathbb{R}^2, \mathbb{R}^2)$. This is wrong, $f'(v)$ is a linear map $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$, for each point $v \in \mathbb{R}^2$. f' can be thought as such a map that associates a linear map to each point $v \in \mathbb{R}^2$, that is $f'(v)$. So, $f' : \mathbb{R}^2 \longrightarrow L(\mathbb{R}^2, \mathbb{R}^2)$, not $f'(v)$.