Analysis II Endsem Part B

Nirjhar Nath
nirjhar@cmi.ac.in
BMC202239

Solution 1:

Let $C \subset Y$ be a connected subset. Assume, to the contrary, that $O_1, O_2 \subset X$ are open subsets that witness the disconnectedness of $f^{-1}(C)$, i.e., $f^{-1}(C) \subseteq O_1 \cup O_2$ and $O_1 \cap f^{-1}(C) \neq \emptyset$ and $O_2 \cap f^{-1}(C) \neq \emptyset$ and $O_1 \cap O_2 \cap f^{-1}(C) = \emptyset$. We observe that $f(O_1), f(O_2)$ are open in Y and that $C \subset f(O_1) \cup f(O_2)$ (find a preimage $c' \in X$ for some $c \in C$, i.e., $f(c') = c$; this $c' \in f^{-1}(C)$ is in O_1 or O_2 and hence $f(c') = c$ is in one of $f(O_1), f(O_2)$ as well). Now, $f(c') \in C \cap f(O_1)$ for an $c' \in O_1 \cap f^{-1}(C)$. Similarly, $f(O_2) \cap C \neq \emptyset$.

Suppose $C \cap f(O_1) \cap f(O_2) \neq \emptyset$. Then consider $c \in C \cap f(O_1) \cap f(O_2)$. We observe that $F_c := f^{-1}(\{f(c)\}) \neq \emptyset$. This means that O_1 and O_2 witness the disconnectedness of F_c, which cannot happen by assumption. So, $C \cap f(O_1) \cap f(O_2) = \emptyset$ and it follows that C is not connected, a contradiction.

Thus, $f^{-1}(C)$ is connected. ■

Solution 4 (ii):

Let $f \in C^0([-\pi, \pi])$ which satisfies for some $C > 0$, $\alpha \in (0, 1]$,

$$|f(x) - f(y)| \leq C|x - y|^\alpha$$

for all $x, y \in [-\pi, \pi]$. We suppose

$$D_n(x) = \frac{1}{\pi} \cdot \frac{\sin((n + 1/2)x)}{\sin(x/2)}.$$

Then the trucated Fourier series is

$$\begin{aligned} S_n(f)(x) &= \int_{-\pi}^{\pi} D_n(y) f(x - y) dy \\ &= \int_0^{\pi} D_n(y) \frac{f(x + y) + f(x - y)}{2} dy. \end{aligned}$$

Let f be defined as above and C be a uniform bound for f. Let $\delta \in (0, \pi)$ be fixed. Then there exists a constant K such that $\frac{1}{\sin(x/2)}$ is uniformly bounded by K on $[\delta, \pi]$. Also, there exists a piecewise linear continuous and 2π periodic function f_ϵ that is bounded by C and satisfies

$$\int_0^{2\pi} |f(x) - f_\epsilon(x)| dx < \frac{\pi\epsilon}{K}.$$

Then,

$$\begin{aligned} &\left| \int_\delta^\pi D_n(y) \frac{f(x + y) + f(x - y) - f_\epsilon(x + y) - f_\epsilon(x - y)}{2} dy \right| \\ &\leq \frac{K}{2\pi} \int_\delta^\pi |f(x + y) - f_\epsilon(x + y)| + |f(x - y) - f_\epsilon(x - y)| \\ &< \frac{K}{2\pi} \cdot \frac{\pi\epsilon}{K} = \frac{\epsilon}{2}. \end{aligned}$$

Using integration by parts, we can show that the integral

$$\int_\delta^\pi D_n(y) \frac{f_\epsilon(x + y) + f_\epsilon(x - y)}{2} dy =: S_n(f_\epsilon)(x)$$

converges to 0 uniformly. Therefore, the integral

$$S_n(f)(x) = \int_\delta^\pi D_N(y)\frac{f(x+y)+f(x-y)}{2}dy$$

also converges to 0 uniformly for fixed $\delta > 0$. Therefore,

$$S_n(f)(x) - f(x) = \int_0^\delta D_n(y)\left[\frac{f(x-y)+f(x+y)}{2} - f(x)\right]dy + E_n(x),$$

where $E_n(x)$ converges to 0 uniformly regardless of the choice of $\delta > 0$.

Thus, given any $\epsilon > 0$, we can choose $\delta > 0$ such that

$$C\int_0^\delta \frac{|\sin((n+1/2)y|}{|\sin(y/2)|}|y|^\alpha dy < \frac{\epsilon}{2},$$

and then choose N large enough such that $E_n(x)$ is uniformly bounded by $\epsilon/2$ for $n > N$. It follows that $|S_n(f)(x) - f(x)| < \epsilon \; \forall \; x$, i.e., the set $\{S_n(f) : n \in \mathbb{Z}\}$ converges uniformly to f. ■

Solution 5:

Let $V = C_{\text{per}}[0, 2\pi]$ be the set of all 2π-periodic functions with the usual sup norm denoted by $||\cdot||$. Let E_x be the dense G_δ-set of continuous 2π-periodic functions in V such that the Fourier series of these functions diverge at x. Let $\{x_i\}$ be a countable set of points in $[0, 2\pi]$ and let

$$E = \bigcap_{i=1}^{n} E_{x_i} \subset V.$$

Then by Baire's theorem, E is also a dense G_δ-set. (Since each E_{x_i} is the countable intersection of dense open sets, so the same is true for E.) Thus, for every $f \in E$, the Fourier series of f diverges at $x_i \; \forall \; i$ (Part A, Problem 6). Define

$$s^*(f; x) := \sup_n s_n(f)(x).$$

Then s^* is a lower semi-continuous function, as it is the supremum of a collection of continuous functions. Therefore, for each f, the set $Q_f := \{x : s^*(f; x) = \infty\}$ is a G_δ-set in $(0, 2\pi)$. If we choose the x_i's such that their union is dense in $(0, 2\pi)$, then we have the following result.

Lemma: The set $E \subset V$ is a G_δ-set such that $\forall \; f \in E$, the set $Q_f \subset (0, 2\pi)$ where its Fourier series diverges, is a dense G_δ-set in $(0, 2\pi)$.

We can now show that Q_f is indeed countable (see the following lemma).

Lemma: In a complete metric space X which has no isolated points, no countable dense set is a G_δ.

Proof: Let $E = \{x_1, x_2, \ldots, x_n\}$ be a countable dense set in X. Assume E is G_δ. Then $E = \cap_{n=1}^\infty W_n$, where each W_n is dense and open. Then by hypothesis,

$$W_n \setminus \bigcup_{i=1}^{\infty}\{x_i\} =: V_n$$

is also open and dense. But then, $\cap_{n=1}^\infty V_n = \emptyset$, which is a contradiction to Baire's theorem.

Thus, Q_f is countable. Therefore, there exists uncountably many continuous functions on $[0, 2\pi]$, whose Fourier series diverge on a dense G_δ subset of $[0, 2\pi]$. ■

Solution 7:

We consider the function $f(x) = x^2$ on $[-\pi, \pi]$ and find its expansion into a trigonometric Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

which is periodic and converges to $f(x)$ in $[-\pi, \pi]$. Since $f(x)$ is even, it is enough to determine the coefficients

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

for $n = 0, 1, 2, \ldots$ because we have,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx = 0 \;\forall\, n \in \mathbb{N}.$$

For $n = 0$, we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{2}{\pi}\int_{0}^{\pi} x^2 dx = \frac{2\pi^2}{3}.$$

For $n \in \mathbb{N}$, we have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx \; dx = \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos nx \; dx = \frac{2}{\pi} \cdot \frac{2\pi}{n^2}(-1)^n = (-1)^n \frac{4}{n^2},$$

because

$$\int x^2 \cos nx \; dx = \frac{2x}{n^2} \cos nx + \left(\frac{x^2}{n} - \frac{2}{n^3}\right) \sin nx + C.$$

Thus,

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left((-1)^n \frac{4}{n^2} \cos nx\right).$$

Putting $x = \pi$, we get

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left((-1)^n \frac{4}{n^2} \cos{(n\pi)}\right) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\left((-1)^n(-1)^n \frac{1}{n^2}\right).$$

Thus, we get

$$\frac{2\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{1}{n^2},$$

and hence,

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6},$$

as required. ■