Analysis II Assignment 1,2

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Solution 1:

Let $d : X \times X \to [0, \infty)$ be a pseudo metric. Define $x \sim y$ if $d(x, y) = 0$. Let $\tilde{X} = X/\sim$ and define $\tilde{d} : \tilde{X} \times \tilde{X} \to [0, \infty)$ by $\tilde{d}([x], [y]) = d(x, y)$. We check the properties of metric.

1. Proof that $\tilde{d}([x], [y]) = 0 \iff [x] = [y]$
   If $\tilde{d}([x], [y]) = 0$, then $d(x, y) = 0$, i.e., $x \sim y$. Also, if $[x] = [y]$, then $x \sim y$, which implies $d(x, y) = 0 = \tilde{d}([x], [y])$.

2. Proof that $\tilde{d}([x], [y]) = \tilde{d}([y], [x])$
   Since d is a pseudo metric, so $\tilde{d}([x], [y]) = d(x, y) = d(y, x) = \tilde{d}([y], [x])$.

3. Proof that $\tilde{d}([x], [y]) \leq \tilde{d}([x], [z])\tilde{d}([z], [y])$
   Similar to the above, $\tilde{d}([x], [y]) = d(x, y) \leq d(x, z) + d(z, y) = \tilde{d}([x], [z])\tilde{d}([z], [y])$.

Therefore, $\tilde{d}$ is a well-defined metric.

Let $x \in A$. Since A is open, there exists an open ball $B_r(x) \subseteq A$ for some $r > 0$. Therefore, if $y \in [x]$, then $d(x, y) = 0 < r \implies y \in A$, which implies $[x] \subseteq A$. Then, $A = \bigcup_{x \in A} [x]$ is a union of equivalence classes.

Take any $[x] \in \pi(A)$ then $x \in A$ and there exists an open ball $B_r(x) \subseteq A$ for some $r > 0$. Now if $[y] \in B_r([x])$ in $\tilde{X}$, then $\tilde{d}([x], [y]) < r$. Therefore, $d(x, y) < r \implies y \in B_r(x) \subset A$. Hence, $\pi(A)$ is open in $\tilde{X}$. ■

Solution 2:

$\mathbb{R}^*$ is the extended real number system $[-\infty, \infty]$. Define $f : \mathbb{R}^* \to [-1, 1]$ by

$$f(x) = \frac{x}{1 + |x|} \quad \forall\, x \in (-\infty, \infty), \quad f(-\infty) = -1, \quad f(\infty) = 1.$$

To show that f is an injection, we need to show that $f(x_1) = f(x_2) \implies x_1 = x_2$. We have,

$$\begin{aligned} & f(x_1) = f(x_2) \\ \implies & \frac{x_1}{1 + |x_1|} = \frac{x_2}{1 + |x_2|} && (1) \\ \implies & x_1 + x_1|x_2| = x_2 + x_2|x_1| \\ \implies & x_1 - x_2 = x_2|x_1| - x_1|x_2| && (2) \end{aligned}$$

From equation (1), we see that x_1 and x_2 must be of same sign or both 0. If $x_1 \geq 0, x_2 \geq 0$, then $x_2|x_1| - x_1|x_2| = x_2x_1 - x_1x_2 = 0$. If $x_1 \leq 0, x_2 \leq 0$, then $x_2|x_1| - x_1|x_2| = x_2(-x_1) - x_1(-x_2) = 0$. Thus $x_2|x_1| - x_1|x_2| = 0$, so equation (2) gives $x_1 = x_2$. Therefore, f is an injection.

Now for $y \in [0, 1)$, $1 - y > 0$ and hence $\frac{y}{1-y} \geq 0$; therefore,

$$f\left(\frac{y}{1-y}\right) = \frac{\frac{y}{1-y}}{1 + \left|\frac{y}{1-y}\right|} = \frac{\frac{y}{1-y}}{1 + \frac{y}{1-y}} = \frac{\frac{y}{1-y}}{\frac{1-y+y}{1-y}} = \frac{\frac{y}{1-y}}{\frac{1}{1-y}} = y$$

and for $y \in (-1, 0)$, $1 + y > 0$ and hence $\frac{y}{1+y} < 0$; therefore,

$$f\left(\frac{y}{1+y}\right) = \frac{\frac{y}{1+y}}{1 + \left|\frac{y}{1+y}\right|} = \frac{\frac{y}{1+y}}{1 - \frac{y}{1+y}} = \frac{\frac{y}{1+y}}{\frac{1+y-y}{1+y}} = \frac{\frac{y}{1+y}}{\frac{1}{1+y}} = y.$$

Also, $f(\infty) = 1$ and $f(-\infty) = -1$. Therefore, f is also a surjection. Hence, f is a bijection.

To show that f is non-decreasing, we need to show that $x_2 \geq x_1 \implies f(x_2) \geq f(x_1)$. We have,

$$f(x_2) - f(x_1) \geq \frac{x_2}{1+|x_2|} - \frac{x_1}{1+|x_1|} = \frac{(x_2 - x_1) + (x_2|x_1| - x_1|x_2|)}{(1+|x_1|)(1+|x_2|)} \tag{3}$$

If $x_2 \geq x_1 \geq 0$, then $x_2|x_1| - x_1|x_2| = x_2x_1 - x_1x_2 = 0$. If $x_2 \geq 0 \geq x_1$, then $x_2|x_1| - x_1|x_2| = x_2(-x_1) - x_1x_2 = -2x_1x_2 \geq 0$. If $0 \geq x_2 \geq x_1$, then $x_2|x_1| - x_1|x_2| = x_2(-x_1) - x_1(-x_2) = 0$. Thus equation (3) implies that $f(x_2) - f(x_1) \geq 0 \iff f(x_2) \geq f(x_1)$. Therefore, f is non-decreasing.

We check that d is a metric, as follows:

1. Proof that $d(x, y) = 0 \iff x = y$
   $d(x, y) = 0 \iff |f(x) - f(y)| = 0 \iff f(x) = f(y) \iff x = y$, since f is an injection.

2. Proof that $d(x, y) = d(y, x)$
   $d(x, y) = |f(x) - f(y)| = |f(y) - f(x)| = d(y, x)$.

3. Proof that $d(x, y) \leq d(x, z) + d(z, y)$
   $d(x, y) = |f(x) - f(y)| \leq |f(x) - f(z)| + |f(z) - f(y)| = d(x, z) + d(z, y)$.

Thus, $d(x, y) = |f(x) - f(y)|$ is a metric.

Since $f : \mathbb{R}^* \to [-1, 1]$ defined by

$$f(x) = \begin{cases} \frac{x}{1+|x|}, \text{ if } x \in (-\infty, \infty) \\ -1, \text{ if } x = -\infty \\ 1, \text{ if } x = \infty \end{cases}$$

is a bijection, therefore $f^{-1} : [-1, 1] \to \mathbb{R}^*$ given by

$$f^{-1}(y) = \begin{cases} \frac{y}{1-|y|}, \text{ if } y \in (-1, 1) \\ -\infty, \text{ if } y = -1 \\ \infty, \text{ if } y = 1 \end{cases}$$

exists and it is clearly continuous. Since $[-1, 1]$ is compact, so its continuous image $(\mathbb{R}^*, d)$ is compact. The open subsets of $(\mathbb{R}^*, d)$ are union of open intervals of the form $(-\infty, a) \cup (b, c) \cup (d, \infty)$. ■

Solution 3:

We prove that d is a metric as follows:

1. Proof that $d(\{x_n\}, \{y_n\}) = 0 \iff \{x_n\} = \{y_n\}$
   $d(\{x_n\}, \{y_n\}) = 0 \iff \sum_{n=1}^{\infty} \frac{1}{2^n}|x_n - y_n| = 0 \iff x_n = y_n \forall\, n \iff \{x_n\} = \{y_n\}$.

2. Proof that $d(\{x_n\}, \{y_n\}) = d(\{y_n\}, \{x_n\})$
   $d(\{x_n\}, \{y_n\}) = \sum_{n=1}^{\infty} \frac{1}{2^n}|x_n - y_n| = \sum_{n=1}^{\infty} \frac{1}{2^n}|y_n - x_n| = d(\{y_n\}, \{x_n\})$.

3. <u>Proof that $d(\{x_n\},\{y_n\}) \leq d(\{x_n\},\{z_n\}) + d(\{z_n\},\{y_n\})$</u>
   We have, $|x_n - y_n| \leq |x_n - z_n| + |z_n - y_n|$. Therefore,

$$\begin{aligned}
d(\{x_n\},\{y_n\}) &= \sum_{n=1}^{\infty} \frac{1}{2^n}|x_n - y_n| \\
&\leq \sum_{n=1}^{\infty} \frac{1}{2^n}|x_n - z_n| + \sum_{n=1}^{\infty} \frac{1}{2^n}|z_n - y_n| \\
&= d(\{x_n\},\{z_n\}) + d(\{z_n\},\{y_n\}).
\end{aligned}$$

Therefore, d is a metric.

Suppose $\{\overline{x_n}\} \subseteq X$ converges to $\{x_n\}_{n=1}^{\infty}$. Then $\forall\ \epsilon > 0$, there exists $N > 0$ such that $\sum_{m=1}^{\infty} \frac{1}{2^k}|x_{m,n} - x_m| < \epsilon\ \forall\ n \geq N$. This implies that $|x_{m,n} - x_m| < \epsilon$ for each $m \in \mathbb{N}$. Therefore, $\{x_{m,n}\}_{m=1}^{\infty}$ converges to x_m for each $m \in \mathbb{N}$.

Conversely, if $\{x_{m,n}\}_{m=1}^{\infty}$ converges to x_m for each $m \in \mathbb{N}$. Thus, for any $\epsilon > 0$, $\exists$ $c_k \in \mathbb{N}$ such that $|x_{k,n} - x_k| < \frac{\epsilon}{2}\ \forall\ n > c_k$. Choose $N \in \mathbb{N}$ such that $\frac{1}{2^{N-2}} < \frac{\epsilon}{2}$. Then,

$$\sum_{k=1}^{N} \frac{1}{2^k}|x_{k,n} - x_k| < \sum_{k=1}^{\infty} \frac{1}{2^k}|x_{k,n} - x_k| < \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{\epsilon}{2} = \frac{\epsilon}{2}.$$

Therefore,

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{1}{2^k}|x_{k,n} - x_k| &= \sum_{k=1}^{N} \frac{1}{2^k}|x_{k,n} - x_k| + \sum_{k=N}^{\infty} \frac{1}{2^k}|x_{k,n} - x_k| \\
&< \frac{\epsilon}{2} + \sum_{k=N}^{\infty} \frac{1}{2^{k-1}} \\
&= \frac{\epsilon}{2} + \frac{1}{2^{N-2}} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

Thus, $\{\overline{x_n}\}$ converges to $\{x_n\}_{n=1}^{\infty}$.

Consider an open ball $B_r(x_n) = \{\sum_{n=1}^{\infty} \frac{1}{2^n}|x_n - y_n| < r\}$ in X. If we take $\{x_n\}$ to be a sequence of rationals such that for some $N \in \mathbb{N}$, $x_n = 0$ for all $n > N$. This sequence is dense in X. Thus, the set of open balls around points of this sequence, i.e., $\{B_r(\{x_n\})\}$ form a countable basis of X. ■

Solution 4:

Given that X and Y are metric spaces and Y is complete. $S \subseteq X$ is dense and $f : S \to Y$ is uniformly continuous. Define the extension $\tilde{f} : X \to Y$ by $\tilde{f}|_S = f$ and for $x \in X \setminus S$, $\tilde{f}(x) = \lim f(s_n)$ where $\{s_n\}$ is any sequence of points in S with $s_n \to x$ (such a sequence exists as S is dense in X).

Claim: $\{f(s_n)\}$ is Cauchy.
Proof: Let $\epsilon > 0$. Since f is uniformly continuous, so $\exists\ \delta > 0$ such that

$$d_X(a,b) < \delta \implies d_Y(f(a), f(b)) < \epsilon.$$

Now since $s_n \to x$, so $\{s_n\}$ is Cauchy and hence $\exists\ N \in \mathbb{N}$ such that $d_X(s_n, s_m) < \delta\ \forall\ n, m \geq N$. By uniform continuity, this implies $d_Y(f(s_n), f(s_m)) < \epsilon\ \forall\ n, m \geq N$. Therefore, $\{f(s_n\})$ is Cauchy.

Since Y is complete, so $\{f(s_n)\}$ converges. Now we prove that $\tilde{f}$ is well-defined, i.e., if $s_n \to x$ and $s'_n \to x$, then $\lim f(s_n) = \lim f(s'_n)$. Let $\{s''_n\} = \{s_1, s'_1, s_2, s'_2, \dots\}$. Let $\epsilon > 0$ be given. Then $\exists\ N_1, N_2 > 0$ such that $d_X(s_n, x) < \epsilon\ \forall\ n \geq N_1$ and $d_X(s'_n, x) < \epsilon\ \forall\ n \geq N_2$. Let $N = \max\{N_1, N_2\}$. Then for $n \geq 2N$, $\lceil \frac{n}{2} \rceil \geq N = \max\{N_1, N_2\}$, so if n is even, then $d_X(s''_n, x) = d_X(s'_{\frac{n}{2}}, x) < \epsilon$ and if n is odd, then $d_X(s''_n, x) = d_X(s_{\frac{n+1}{2}}, x) < \epsilon$. Thus, $s''_n \to x$. Thus, $\lim f(s''_n)$ exists and since $\{f(s_n)\}$ and $\{f(s'_n)\}$ are subsequences of $\{f(s''_n)\}$, so $\lim f(s_n) = \lim f(s''_n) = \lim f(s'_n)$.

Now we prove that $\tilde{f}$ is continuous on $X \setminus S$. ($\tilde{f}$ is continuous on S as $\tilde{f}|_S = f$ and $f : S \to Y$ is uniformly continuous, hence continuous.) Let $\epsilon > 0$. Since f is uniformly continuous, $\exists\ \delta > 0$ such that

$$d_X(a, b) < \delta \implies d_Y(f(a), f(b)) < \frac{\epsilon}{3}.$$

Let $x, y \in X \setminus S$ with $d_X(x, y) < \frac{\delta}{3}$. Since S is dense in X, $\exists$ sequences $\{x_n\}$ and $\{y_n\}$ in S with $x_n \to x$ and $y_n \to y$. Thus, $\exists\ N_1 \in \mathbb{N}$ such that $d_X(x_n, x) < \frac{\delta}{3}$ and $d_X(y_n, y) < \frac{\delta}{3}$ $\forall\ n \geq N_1$. Thus, for $n \geq N_1$,

$$d_X(x_n, y_n) \leq d_X(x_n, x) + d_X(x, y) + d_X(y, y_n) < \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta.$$

Therefore, by uniform continuity of f, $d_Y(f(x_n), f(y_n)) < \frac{\epsilon}{3}$. Also, since $f(x_n) \to \tilde{f}(x)$ and $f(y_n) \to \tilde{f}(y)$, so $\exists\ N_2 \in \mathbb{N}$ such that $d_Y(f(x_n), \tilde{f}(x)) < \frac{\epsilon}{3}$ and $d_Y(f(y_n), \tilde{f}(y)) < \frac{\epsilon}{3}$ $\forall\ n \geq N_2$. Since $\tilde{f}|_S = f$ and $\{x_n\}$, $\{y_n\}$ are sequences in S, so $\tilde{f}(x_n) = f(x_n)$ and $\tilde{f}(y_n) = f(y_n)$. Thus, for $n \geq \max\{N_1, N_2\}$,

$$d_Y(\tilde{f}(x), \tilde{f}(y)) \leq d_Y(\tilde{f}(x), \tilde{f}(x_n)) + d_Y(\tilde{f}(x_n), \tilde{f}(y_n)) + d_Y(\tilde{f}(y_n), \tilde{f}(y)) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Therefore, $\tilde{f}$ is continuous on X.

Now we just have to prove that $\tilde{f}$ is unique. Let $\tilde{f}'$ be another continuous extension of f on X. Then for $x \in X \setminus S$, let $\{s_n\} \subseteq S$ such that $s_n \to x$. Then since $\tilde{f}'|_S = f$, so

$$\tilde{f}'(x) = \lim \tilde{f}'(s_n) = \lim f(x_n) = \tilde{f}(x).$$

Thus, $\tilde{f}$ is unique. ■

Solution 5:

Given, $A, B \subset \mathbb{R}^n$. Define $A + B := \{a + b \mid a \in A, b \in B\}$.

We shall prove that if A and B are open, then $A + B$ is open. Let $c \in A + B$, so $c = a + b$ for some $a \in A$ and $b \in B$. Since A and B are open, so $\exists\ \epsilon_1, \epsilon_2 > 0$ such that $B_{\epsilon_1}(a) \subseteq A$ and $B_{\epsilon_2}(b) \subseteq B$. Let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. Now let $x \in B_\epsilon(c)$ be arbitrary. Then,

$$||c - x|| = ||(a + b) - x|| < \epsilon \implies ||a - (x - b)|| < \epsilon \leq \epsilon_1.$$

Therefore, $x - b \in A$ and hence, $x = (x - b) + b \in A + B$. Thus, $B_\epsilon(c) \subseteq A + B$ and hence, $A + B$ is open.

If A and B are closed, then $A + B$ is not necessarily closed. Consider $A = \mathbb{N}$ and $B = \left\{-n + \frac{1}{n^2} \mid n \in \mathbb{N}\right\}$. Then $A + B = \{\frac{1}{n^2} \mid n \in \mathbb{N}\}$. The sequence $\{\frac{1}{n^2}\}$ converges to 0, so 0 is a limit point of $A + B$. But $0 \notin A + B$. Thus, $A + B$ does not contain all its limit points and hence $A + B$ is not closed. ■

Solution 6:

We show that d is a metric as follows:

1. Proof that $d(m,n) = 0 \iff m = n$
   $d(m,n) = 0 \iff \left|\frac{1}{m} - \frac{1}{n}\right| = 0 \iff \frac{1}{m} = \frac{1}{n} \iff m = n.$

2. Proof that $d(m,n) = d(n,m)$
   $d(m,n) = \left|\frac{1}{m} - \frac{1}{n}\right| = \left|\frac{1}{n} - \frac{1}{m}\right| = d(n,m).$

3. Proof that $d(m,n) \leq d(m,k) + d(k,n)$
   $d(m,n) = \left|\frac{1}{m} - \frac{1}{n}\right| \leq \left|\frac{1}{m} - \frac{1}{k}\right| + \left|\frac{1}{k} - \frac{1}{n}\right| = d(m,k) + d(k,n).$

Therefore, d is a metric.

Consider $S \subseteq \mathbb{N}^* = \mathbb{N} \cup \{\infty\}$.

Claim: If $\infty \notin S$, then S is open.
Proof: Let $x \in S$. Then for $n > x$,

$$d(x,n) = \left|\frac{1}{x} - \frac{1}{n}\right| = \frac{1}{x} - \frac{1}{n} \geq \frac{1}{x} - \frac{1}{x+1}$$

and for $n < x$,

$$d(x,n) = \left|\frac{1}{x} - \frac{1}{n}\right| = \frac{1}{n} - \frac{1}{x} \leq \frac{1}{x-1} - \frac{1}{x}.$$

Therefore, for $r < \frac{1}{x-1} - \frac{1}{x}$, $B_r(x) = \phi \subseteq S$, and hence S is open.

If $\infty \in S$, then if ∞ is an interior point of S, $\exists$ an open ball $B_r(\infty) \subseteq S$ for some $r > 0$. Now for any $N \in B_r(\infty)$,

$$d(N,\infty) = \left|\frac{1}{N} - \frac{1}{\infty}\right| < r \implies \frac{1}{N} < r \implies \frac{1}{n} < r \; \forall \, n \geq N.$$

Thus, for ∞ to be an interior point of S, $\exists \; N \in \mathbb{N}$ such that $n \geq N \implies n \in S$. Thus, open sets of $\mathbb{N}^*$ are (all sets not containing ∞) $\cup$ (sets that contain all $n \geq N$ for some $N \in \mathbb{N}^*$).

$\mathbb{N}$, with respect to the restricted metric is not complete. Consider the sequence $\{x_n\} = \{1, 2, \dots\}$ given by $x_n = n$. Now for a given $\epsilon > 0$, let $N > \frac{2}{\epsilon}$. Then for $n, m > N$,

$$d(x_n, x_m) = d(n,m) = \left|\frac{1}{n} - \frac{1}{m}\right| < \frac{1}{m} + \frac{1}{n} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus, $\{x_n\}$ is Cauchy. But $\forall \; k \in \mathbb{N}$, $d(x_n, k) \to \frac{1}{k}$ as $n \to \infty$. Therefore, the sequence $\{x_n\} = n$ is Cauchy but not convergent. Hence, $(\mathbb{N}, d)$ is not complete.

Suppose $f : \mathbb{N}^* \to \mathbb{N}^*$ is continuous. Then for every open set U in $\mathbb{N}^*$, $f^{-1}(U)$ is open in $\mathbb{N}^*$. Let $f(\infty) = x$. Then for all open sets $S \subseteq \mathbb{N}^*$ not containing x, $f^{-1}(S)$ does not contain ∞ and hence is open. If $f^{-1}(x)$ contains ∞, then for it to be open, it should contain all $n \geq N$ for some $N \in \mathbb{N}^*$; thus $f(n) = x \; \forall \; n \geq N$. Therefore, f is continuous if and only if $\exists \; N \in \mathbb{N}$ such that $f(n) = f(\infty) \; \forall \; n \geq N$. ■

Solution 7:

Given $g \in C[0,1]$. The map $I : C[0,1] \to \mathbb{R}$ is defined by

$$I(f) = \int_0^1 f(x)g(x)dx.$$

We shall prove that I is in fact uniformly continuous and hence continuous. Let $\epsilon > 0$ be given. Let $\int_0^1 |g(x)|dx = k$. For $f_1, f_2 \in C[0,1]$, if $||f_1 - f_2||_\infty < \frac{\epsilon}{k}$, then we can choose $\delta = \frac{\epsilon}{k}$ such that the following holds:

$$|f_1(x) - f_2(x)| < \delta \implies |I(f_1) - I(f_2)| < \epsilon.$$

We prove this as follows:

$$\begin{aligned}
|I(f_1) - I(f_2)| &= \left| \int_0^1 f_1(x)g(x)dx - \int_0^1 f_2(x)g(x)dx \right| dx \\
&= \left| \int_0^1 (f_1(x) - f_2(x))g(x) \right| dx \\
&\leq \int_0^1 |(f_1(x) - f_2(x))g(x)|dx \\
&= \int_0^1 |f_1(x) - f_2(x)| \cdot |g(x)|dx \\
&< \int_0^1 \frac{\epsilon}{k} \cdot kdx = \epsilon.
\end{aligned}$$

Thus, I is uniformly continuous and hence continuous. ■

Solution 8:

Given $g \in C[0,1]$. Let $I_x(f) = \int_0^x f(t)g(t)dt$. We need to show that the set $S = \{f \in C[0,1] \mid I_x(f) \leq x\}$ is closed with respect to the $|| \cdot ||_\infty$ norm. We want to show that S is closed. It suffices to prove that $\overline{S} = C[0,1] \setminus S$ is open in $C[0,1]$. Consider a function F in $\overline{S}$. Then, $\exists\, y \in [0,1]$ such that $I_y(F) > y$. Let $\int_0^y F(t)dt = k$. Choose $\epsilon < \frac{1}{k}(I_y(F) - y)$. Then if $||F - F'||_\infty < \epsilon$, i.e., $F'(x) > F(x) - \epsilon \;\forall\, x \in [0,1]$, we have,

$$\begin{aligned}
I_y(F') &> \int_0^y (F(t) - \epsilon)g(t)dt \\
&= I_y(F) - \epsilon \int_0^y g(t)dt \\
&> I_y(F) - \frac{1}{k}(I_y(F) - y)k \\
&= I_y(F) - I_y(F) + y \\
&= y.
\end{aligned}$$

Thus, $I_y(F') = \int_0^y F'(t)g(t)dt > y$ and hence, $B_\epsilon(F(x)) \subseteq \overline{S}$. Therefore, $\overline{S}$ is open in $C[0,1]$ and hence S is closed in $[0,1]$. ■

Solution 9:

Since $\overline{A}$ is closed, so $\overline{A}^c$ is open. Therefore,

$$A \subseteq \overline{A} \implies A^c \supseteq (\overline{A})^c \implies \text{Int}(A^c) \supseteq (\overline{A})^c.$$

Also, since $\mathrm{Int}(A^c)$ is open, so $(\mathrm{Int}(A^c))^c$ is closed. Therefore,

$$\mathrm{Int}(A^c) \subseteq A^c \implies (\mathrm{Int}(A^c))^c \supseteq A \implies (\mathrm{Int}(A^c))^c \supseteq \overline{A} \implies \mathrm{Int}(A^c) \subseteq (\overline{A})^c.$$

Therefore, $\mathrm{Int}(A^c) = (\overline{A})^c$. ■

Solution 10:

(i) f is continuously differentiable on $\mathbb{R}$ and $f_n(x) = n\left(f(x+\frac{1}{n}) - f(x)\right)$. Now, f is also continuous and since $[a, b]$ is compact, so f is also uniformly continuous on $[a, b]$. Thus, given any $\epsilon > 0$, $\exists\ \delta > 0$ such that

$$|x - y| < \delta \implies |f'(x) - f'(y)| < \epsilon. \tag{4}$$

Let $\epsilon > 0$ be fixed, by Mean Value theorem,

$$f_n(x) = f'(c_x) \text{ for some } c_x \in \left[x, x + \frac{1}{n}\right]. \tag{5}$$

For $n > \frac{1}{\delta}$, $|x - c_x| < \frac{1}{n} < \delta$. Thus, substituting $y = c_x$ in equation (4), we have $|f'(x) - f'(c_x)| = |f(c_x) - f'(x)| < \epsilon$. Now, using equation (5), we have $|f_n(x) - f'(x)| < \epsilon\ \forall\ x \in [a, b]$. Therefore, we have showed that for any given $\epsilon > 0$, $\exists\ N = \frac{1}{\delta} > 0$ such that $|f_n(x) - f'(x)| < \epsilon\ \forall\ n \geq N$. Thus, f_n uniformly converges to f' on any finite interval $[a, b]$.

(ii) $A_n \in M_{n\times m}$ is a function from $\mathbb{R}^m$ to $\mathbb{R}^n$. Given, $A_n \to A$ pointwise. Therefore, $A_n e_i \to Ae_i$, i.e., the i^{th} column of A_n converges to the i^{th} column of $A\ \forall\ i$. Thus, given any $\epsilon > 0$, $\exists\ N > 0$ such that $||(A_n - A)e_i|| < \epsilon\ \forall\ n \geq N$ and $\forall\ i$. Therefore, $A \in M_{n\times m}$. Consider a compact subset $K \subseteq \mathbb{R}^m$ and a column vector $v = \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} \in K \subseteq \mathbb{R}^m$. Then,

$$||(A_n - A)v|| = \left|\left|\sum_{i=1}^{m} v_i(A_n - A)e_i\right|\right| \leq \sum_{i=1}^{m} v_i||(A_n - A)e_i|| < \left(\sum_{i=1}^{m} v_i\right)\epsilon$$

Consider the mapping $f : K \to \mathbb{R}^m$ such that $v = \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} \mapsto \sum_{i=1}^{m} v_i$. Clearly, f is continuous and since K is compact, it attains a maximum on K. Let the maximum value attained be M. Therefore, $||A_n - A||_\infty < M\epsilon\ \forall\ n \geq N$. Also, for each i, $||(A_n - A)e_i|| < \frac{\epsilon}{M}$. Thus, given $\epsilon > 0$, we can choose $\delta = \frac{\epsilon}{M}$ such that $||A_n - A||_\infty < M \cdot \frac{\epsilon}{M} = \epsilon$. Therefore, $A_n \to A$ uniformly on compact subsets of $\mathbb{R}^m$.

(iii) f is a continuous function on $[0, 1]$. The function f_n on $[0, 1]$ is defined by

$$f_n(x) = f\left(\frac{k-1}{n}\right), \text{ if } \frac{k-1}{n} \leq x < \frac{k}{n}, k = 1, 2, \ldots, n$$

and $f_n(1) = f(1)$. Since $[0, 1]$ is compact, so f is also uniformly continuous on $[0, 1]$. Therefore, for any $\epsilon > 0$, $\exists\ \delta > 0$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

Choose n such that $n > \frac{1}{\delta}$. If $\frac{k-1}{n} \leq x < \frac{k}{n}$, then by the choice of δ, we have $\left|x - \frac{k-1}{n}\right| < \frac{1}{n} < \delta$. By uniform continuity, we have

$$\left|x - \frac{k-1}{n}\right| < \delta \implies \left|f(x) - f\left(\frac{k-1}{n}\right)\right| < \epsilon \implies |f(x) - f_n(x)| < \epsilon$$

$\forall\ x \in [0,1)$ Also, for $x = 1$, $f_n(1) = f(1)$ and hence $|f_n(1) - f(1)| = 0 < \epsilon$. Therefore, given $\epsilon > 0$, $\exists\ N = \frac{1}{\delta} > 0$ such that $|f_n(x) - f(x)| < \epsilon\ \forall\ n > N$ and $x \in [0,1]$. Therefore, $f_n \to f$ as $n \to \infty$ uniformly on $[0,1]$. ■