CHENNAI MATHEMATICAL INSTITUTE

B.Sc. Analysis-2

Mid-term Examination, 2023, Aug-Nov

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Part A

The problems in this section are either already discussed in the class or a slight variation of that. Give the required proof completely with all details. You can score up to a maximum of 50 marks from this section.

1. Prove either that l^2 is complete or that the vector spaces $c = \{\{x_n\}_{n=1}^{\infty} : x_n \in \mathbb{C},\ \{x_n\}_{n=1}^{\infty}$ is convergent$\}$ and $c = \{\{x_n\}_{n=1}^{\infty} : x_n \mapsto 0\}$ are complete 10

2. Let X be a complete metric space and $\mathcal{F}$ a family of continuous real valued functions on X, which is unbounded on every open ball B, that is $sup_{f\in\mathcal{F},x\in B}|f(x)| = \infty$, then show that the set $S = \{x \in X : sup_{f\in\mathcal{F}}|f(x)| = \infty\}$ is dense in X. 10

3. Let K be a compact subset of a metric space X. Let $\{B_n\}_{n=1}^{\infty}$ be an open cover for K. Show that there exists an $\epsilon > 0$ such that any open ball of radius ϵ centered at any point $x \in K$ is contained in one of the B_ns. 10

4. Show that every metric space can be isometrically embedded in its space of bounded uniformly continuous real functions. 10

5. Let X be a complete metric space. Let $T : X \mapsto X$ be continuous and T^n is a contraction for some $n \geq 1$. Show that T has a unique fixed point. 10

6. For $f : [0,1] \mapsto \mathbb{R}$, define

$$(D^+f)(a) = \limsup_{x\mapsto a^+} \frac{f(x) - f(a)}{x - a}.$$

Prove that for each $a \in [0,1]$ the set $\{f \in C[0,1] : (D^+f)(a) = \infty\}$ is a dense G_δ subset. (A set is said to be G_δ if it is countable intersection of open sets.) 20

Part B

You may use any result proved in the class or in assignments. You can score up to a maximum of 75 marks from this section.

1. Let $X = \mathbb{R}^2$ and d_2 be the usual metric on X. Define for $x, y \in X$,

$$\begin{aligned} \rho(x,y) &= d_2(x,y), \quad \text{if } x = \lambda y \text{ for some } \lambda \in \mathbb{R}, \\ &= d_2(x,0) + d_2(0,y), \quad \text{otherwise.} \end{aligned}$$

Prove that ρ is a metric which generates stronger topology than d_2. Also show that the closed unit ball (in the metric d) is not compact in (X, ρ). Is it separable? 40

2. Let X be a separable metric space and $S \subseteq X$ be an uncountable subset. Show that there exists an $x \in S$ such that for all neighbourhood $U \ni x$, $U \cap S$ is uncountable. 15

3. For $n \in \mathbb{Z}, \alpha > 0$ define

$U_\alpha(n) = \{x \in \mathbb{R} : d(nx, \mathbb{Z}) < n^{-\alpha}\}$; $Y_\alpha = \{x \in \mathbb{R} : x \text{ belongs to } U_\alpha(n) \text{ for infinitely many } n\}$.

Show that Y_α is a G_δ subset of $\mathbb{R}$ and $Y = \cap_{\alpha \in (0,\infty)} Y_\alpha$ is dense G_δ subset of $\mathbb{R}$. 25

4. Let $X = \{\{x_n\}_{n=1}^\infty : 0 \leq x_n \leq 1, \ \forall n \in \mathbb{N}\}$ with $d(\{x_n\}, \{y_n\}) = \sum_{n=1}^\infty \frac{1}{2^n} |x_n - y_n|$. Show that (X, d) is compact. 12

5. Let d be the usual metric on $X = [0, 1)$ given by $d(x, y) = |x - y|$. Let

$$D(x,y) = \left|\frac{x}{1-x} - \frac{y}{1-y}\right|.$$

Show that D is a metric and (X, d) is homeomorphic to (X, D). Determine the completeness of (X, d) and (X, D). 25

6. Let X, Y be compact metric spaces. Show that a $f : X \mapsto Y$ is continuous if and only if the graph $\{(x, f(x)) : x \in X\} \subseteq X \times Y$ is closed. 12

7. Prove that that a metric space X is compact if and only if all (real valued) continuous functions are bounded. 10

8. Let $f : X \mapsto Y$ be continuous map onto Y and X be compact. Also $g : Y \mapsto Z$ is such that $g \circ f$ is continuous. Show g is continuous. 12

9. Let X be a metric space and $\{K_n\}_{n=1}^\infty$ is a decreasing sequence of compact subsets of X. Let $f : X \mapsto X$ be a continuous function, show that

$$f\left(\bigcap_{n=1}^{\infty} K_n\right) = \bigcap_{n=1}^{\infty} f(K_n).$$

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