Analysis II

Quiz 4

Attempt all 4 problems. Make sure to properly state the theorems you are using. **Total Marks=40**

Problem 1

Let (X, d) be a complete metric space. Provide an example of a descending countable collection of closed, nonempty subsets of X whose intersection is empty. Does this contradict the Cantor Intersection Theorem? Can you provide an example if (X, d) is assumed to be compact? [10]

Problem 2

1. Let (X, d) be a complete space. Show a perfect subset of X is uncountable. Does the conclusion still hold if X is not assumed to be complete? [5]
2. Does there exists a continuous function $f : [0, 1] \to \mathbb{R}$ which is not monotone on any subinterval? [5]

Problem 3

1. State Banach Contraction Principle. [2]
2. Let X be a compact metric space and T be a map from (X, ρ) to itself such that

$$\rho(T(u), T(v)) < \rho(u, v) \ \forall \ u, v \in X$$

unique
fixed

   Prove that T has a unique fixed point. Does this conclusion still hold if (X, ρ) is assumed to be complete and not compact? [8]

Problem 4

Let f be a real-valued function on a metric space X. Show that the set of points at which f is continuous is the intersection of a countable collection of open sets. Conclude that there is not a real-valued function on $\mathbb{R}$ that is continuous just at the rational numbers. [10]