ANALYSIS 2 - QUIZ 1

September 6, 2023

This quiz has **two questions**, and will be for **30 minutes**.

Total Points: 30 **Maximum Points: 20**

1. (15 points) Recall $\mathcal{C}[0,1]$ to be the set of continuous functions $f : [0,1] \to \mathbb{R}$. For $\phi \in \mathcal{C}[0,1]$, define the function $\rho_\phi : \mathcal{C}[0,1] \times \mathcal{C}[0,1] \to \mathbb{R}$ as below:

$$\rho_\phi(f,g) = \int_0^1 |f(x) - g(x)|\phi(x)dx \tag{1}$$

   (a) (5 points) Show that ρ_ϕ is a metric if $\phi(x) > 0$ for all $x \in [0,1]$.

   (b) (6 points) Suppose now that ρ_ϕ is a metric for some ϕ. Show there exists no interval $[a,b] \subseteq [0,1]$ such that $\phi(x) \leq 0$ for all $x \in [a,b]$. Hint: Suppose ρ_ϕ is a metric for some ϕ which is non-positive in an interval in $[0,1]$, and lead this to a contradiction.

   (c) (4 points) Show ρ_ϕ and ρ_{ϕ_0} are equivalent metrics if $\phi(x) > 0$ for all $x \in [0,1]$, with $\phi_0(x) = 1$ for all $x \in [0,1]$.

2. (15 points) Let d be a metric on a set X. Assume that it satisfies the *ultrametric* inequality:

$$d(x,z) \leq \max\{d(x,y), d(y,z)\} \tag{2}$$

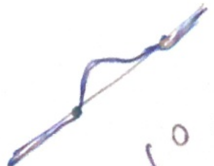
   One example of a ultrametric space is the rationals $\mathbb{Q}$ equipped with the p-adic metric. Prove the following statements about ultrametric spaces:

   (a) (5 points) Every triple of points forms an isosceles triangle. In other words, equality holds in inequality (2) whenever $d(x,y) \neq d(y,z)$.

   (b) (5 points) Any ball in (X,d) is both closed and open.

   (c) (5 points) Every point in a ball is the centre of the ball. In other words, for all $y \in X$, $\epsilon > 0$ and $x \in B(y,\epsilon)$, $B(x,\epsilon) = B(y,\epsilon)$.

All The Best!



$(x) = \begin{cases} 0, & x \in [0,a) \\ \sin\left(\frac{6x}{\pi} - \right. \end{cases}$