**CHENNAI MATHEMATICAL INSTITUTE**

**Analyisis- 2, 2023**

*Assignment 2*

1. Let $f, g : [0,1] \mapsto \mathbb{R}$ be continuous functions satisfying

$$\int_0^1 f(t)t^n dt = \int_0^1 g(t)t^n dt, \quad \forall \text{ integers } n \geq 0.$$

   Show that $f = g$.

2. Let $X$ be a connected metric space. $f : X \mapsto \mathbb{R}$ be a function satisfying, for all $x \in X$ there exists a open set $U_x \ni x$ such that $f$ is constant on $U_x$. Show that $f$ is constant on $X$.

3. Is the set $\{z : z = \exp(i \sin t),\ t \in \mathbb{R}\} \subseteq \mathbb{C}$. connected? Prove your answer.

4. Let $\mathcal{B}$ be the collection of continuous functions on $\mathbb{R}$ which are eventually constant, that is there exists a $M > 0$ such that $f(x)$ is constant on the set $\{x; |x| > M\}$. Verify that $\mathcal{B}$ is a an algebra containing constants and separates points of $\mathbb{R}$. But show that there are continuous functions which can not be approximated uniformly by functions in $\mathcal{B}$. Does this contradict Stone-Weierstrass theorem. Explain your answer.

5. Let $\mathcal{F} \subseteq C([0,1])$ be a family such that

   (i) $f'(t)$ exists for all $t \in (0,1)$ and $f \in \mathcal{F}$,

   (ii) $\sup_{f \in \mathcal{F}} |f(0)| < \infty$,

   (iii) $\sup_{t \in (0,1), f \in \mathcal{F}} |f'(t)| < \infty$.

   Show that the closure of $\mathcal{F}$ is compact.

6. Show that

$$\frac{(3x^2 - \pi^2)}{12} = \sum_{n=1}^{\infty} (-1)^n n^{-2} \cos nx$$

   for all $x \in [-\pi, \pi]$.