CHENNAI MATHEMATICAL INSTITUTE

Analyisis- 2, 2023

Assignment 1

1. Let $d : X \times X \mapsto [0, \infty)$ be a pseudo metric. Define $x \sim y$ if $d(x, y) = 0$. Let $\tilde{X} = X/\sim$ and define $\tilde{d} : \tilde{X} \times \tilde{X} \mapsto [0, \infty)$ by $d([x], [y]) = d(x, y)$. Show that $\tilde{d}$ is a well-defined metric. Prove that a open subset A of (X, d) (defined through the pseudo metric in a similar way to metric) is a union of equivalence classes and $\pi(A)$ is open in $\tilde{X}$, where $\pi : X \mapsto \tilde{X}$ is defined by $\pi(x) = [x]$.

2. Let $\mathbb{R}^*$ be the extended real number system $[-\infty, \infty]$. Define $f : \mathbb{R}^* \mapsto [-1, 1]$ by

$$f(x) = \frac{x}{1 + |x|} \quad \forall x \in (-\infty, \infty), \quad f(-\infty) = -1, \qquad f(\infty) = 1.$$

   Show that f is a bijection and non-decreasing. Prove that $d(x, y) = |f(x) - f(y)|$ is a metric. Describe the open subsets of $(\mathbb{R}^*, d)$. Is it compact?

3. Let $X = \{\{x_n\}_{n=1}^{\infty} : 0 \leq x_n \leq 1, \ \forall n \in \mathbb{N}\}$. Define $d(\{x_n\}, \{y_n\}) = \sum_{n=1}^{\infty} \frac{1}{2^n}|x_n - y_n|$. Show that d is metric. Show that a sequence $\{\overline{x}_n\} \subseteq X$, with $\overline{x}_n = \{x_{m,n}\}_{m=1}^{\infty}$ converges if and only if $\{x_{m,n}\}_{n=1}^{\infty}$ converges for each $m \in \mathbb{N}$. Describe the open subsets of X.

4. Let X, Y be metric spaces. Let $S \subseteq X$ be dense and $f : S \mapsto Y$ be uniformly continuous. Show that f can be extended uniquely as a continuous function to $X \mapsto Y$.

5. Let $A, B \subset \mathbb{R}^n$ and define

$$A + B = \{a + b; \ a \in A, \ b \in B\}.$$

   If A and B are open is $A + B$ open? If A and B are closed is $A + B$ closed?

6. Let $\mathbb{N}^* = \mathbb{N} \cup \{\infty\}$. Assume $\frac{1}{\infty} = 0$. Define

$$d(m, n) = |\frac{1}{m} - \frac{1}{n}|.$$

   Verify d is a metric. Describe the open sets. Is $\mathbb{N}$, with respect to the restricted metric, a complete metric space? When is a function $f : \mathbb{N}^* \mapsto \mathbb{N}^*$ continuous?

7. Let $g \in C([0, 1])$. Prove that the map $I : C([0, 1]) \mapsto \mathbb{R}$ defined by

$$I(f) = \int_0^1 f(x)g(x)dx$$

   is continuous.

8. Let $g \in C[0,1]$. Show that the set $\{f \in C[0,1] : \int_0^x f(t)g(t)dt \leq x\}$ is closed with respect to $\|\cdot\|_\infty$ norm.

9. Show that $Int(A^c) = (\overline{A})^c$.

10. (i) Let f be continuously differentiable on $\mathbb{R}$. Let $f_n(x) = n\left(f\left(x+\frac{1}{n}\right) - f(x)\right)$. Prove that f_n converges uniformly to f' on any finite interval $[a,b]$.

    (ii) Let $A_n \in M_{n\times m}$ considered as function from $\mathbb{R}^m$ to $\mathbb{R}^n$. Suppose A_n converges to A pointwise, Show that that $A \in M_{n\times m}$. Also show $A_n \to A$ uniformly on compact subsets of $\mathbb{R}^m$.

    (iii) Let f be a continuous function on $[0,1]$. Define a function f_n on $[0,1]$ by

$$f_n(x) = f\left(\frac{k-1}{n}\right), \text{ if } \frac{k-1}{n} \leq x < \frac{k}{n} \; k = 1, 2, \cdots n,$$

    and $f_n(1) = f(1)$. Show that f_n converges uniformly to f as $n \to \infty$ on $[0,1]$.