Analysis I, Mid-Semester Exam
28 September, 2022
Total Points: 30, Duration: 2.5 hours

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ denote the sets of positive integers, integers, rational numbers and real numbers, respectively.
- When considered as a metric space, $\mathbb{R}$ has the usual euclidean metric: $d(x,y) = |x-y|$ for real numbers x, y.

1. (1+1+3+3 points) State true or false. You have to justify your answers for full marks.
   (a) Let $\{x_n\}$ be a bounded sequence of real numbers such that $|x_n| \leqslant |x_{n+1}|$ for all $n \geqslant 1$. Then $\{x_n\}$ is convergent.
   (b) Let $\{x_n\}$ be a sequence of real numbers. Suppose that there exists a real number x such that $|x_{n+1} - x| < |x_n - x|$ for all $n \geqslant 1$. Then $\{x_n\}$ converges to x.
   (c) The set $S = \left\{\frac{m}{100^n} \mid m, n \in \mathbb{Z}\right\}$ is dense[1] in $\mathbb{R}$.
   (d) Consider $\mathbb{Q}$ as a metric space with the usual euclidean metric[2] and let $E \subset \mathbb{Q}$ be the subset defined as: $E = \left\{q \in \mathbb{Q} \mid 0 \leqslant q^2 < 5\right\}$. Then E is closed in $\mathbb{Q}$ and bounded, but not compact.

2. (4 points) Show that every open set in $\mathbb{R}$ is a union of an at most countable collection of disjoint open intervals.
   (*Hint:* Use the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$.)

3. (3+3 points) Determine if the following sets of real numbers are bounded above or bounded below and where applicable, find their supremum/infimum.
   (a) $S_1 = \left\{x \in \mathbb{R} \mid x \neq 0, x < \frac{1}{x}\right\}$.
   (b) $S_2 = \left\{x \in \mathbb{R} \mid x^2 \leqslant x + 1\right\}$.

4. (4+4+4 points)
   (a) Let $\{x_n\}$ be a bounded sequence of real numbers such that $x_n \leqslant x_{n+1}$ for all $n \geqslant N_0$ for some positive integer N_0. Show that $\{x_n\}$ is convergent.
   (b) Define a sequence $\{a_n\}$ of real numbers as follows:
   $$a_1 = 1, \text{ and } a_{n+1} = \sqrt{2 + a_n}, \text{ for } n \geqslant 1.$$
   Is the sequence $\{a_n\}$ convergent? If so, find its limit.
   (*Hint:* Show that $a_n < a_{n+1}$ for all $n \geqslant 1$.)
   (c) Define a sequence $\{b_n\}$ of real numbers as follows[3]
   $$b_n = \sum_{i=0}^{n-1} \frac{1}{i!}, \text{ for } n \geqslant 1.$$
   Show that the sequence $\{b_n\}$ is convergent and that its limit belongs to the interval $(2, 3)$.
   (*Hint:* You may use the following formula, without proof: $1 + \frac{1}{2} + \ldots + \frac{1}{2^{n-1}} = \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}}$, for every $n \geqslant 1$.)

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[1] A subset $S \subset \mathbb{R}$ is *dense* in $\mathbb{R}$ if every nonempty open interval (a, b) in $\mathbb{R}$ contains an element of S.
[2] $d(p, q) = |p - q|$ for $p, q \in \mathbb{Q}$.
[3] We set $0! = 1$.