Algebra III, Mid-Semester Exam
3 October, 2023
Total Points: 60, Duration: 2.5 hours

- All rings in this exam are commutative and have a multiplicative identity denoted by 1.
- $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{C}$ denote the rings of integers, rational numbers and complex numbers, respectively.

1. (5+5 points) Determine true or false, with justification.
   (a) Let R be a ring such that for every element $x \in R$, there exists an integer $n \geqslant 2$ (depending on x) such that $x^n = x$. Then every prime ideal of R is maximal.
   (b) Let R be a ring such that every prime ideal of R is maximal. Then R is a field.
2. (5 points) Determine the number of ideals, prime ideals and maximal ideals of the ring $\mathbb{Z}/124\mathbb{Z}$.
3. (10 points) Let I, J be ideals in a ring R such that $I + J = R$. Show that $R/(I \cap J) \cong R/I \times R/J$. ($\cong$ denotes ring isomorphism and $\times$ denotes the direct product of rings.)
4. (10 points) Let L be a complex line $\{ax + by + c = 0\}$ in $\mathbb{C}^2$ (at least one of a or b is nonzero) and let C be an algebraic curve $\{f(x, y) = 0\}$, where $f \in \mathbb{C}[x, y]$ is an irreducible polynomial of degree $d > 0$. Show that $C \cap L$ contains at most d points unless $C = L$.
5. (5+5 points) A ring R is said to be *local* if R contains exactly one maximal ideal.
   (a) Show that a ring is local if and only if the set of non-units forms an ideal.
   (b) Suppose that $\mathfrak{m}$ is a maximal ideal of a ring R. If every element of $1 + \mathfrak{m} := \{1 + a \mid a \in \mathfrak{m}\}$ is a unit show that R is local.
6. (15 points) Suppose that R is an integral domain in which every prime ideal is principal. Show that every ideal of R is principal. (Hint: Consider the collection of all non-principal ideals of R.)