Algebra III, Final Exam
28 November, 2023, 09:30-12:30
Total Points: 80, Duration: 3 hours

- $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ denote the rings of integers, rational numbers, real numbers and complex numbers, respectively. For a prime p, $\mathbb{F}_p$ denotes the field $\mathbb{Z}/p\mathbb{Z}$.
- Draw a table like this at the TOP of PAGE 1 of your solutions. Put the numbers of the two questions you want to be graded among questions 5, 6, 7 in the two empty boxes in the first row.

1	2	3	4		

1. (15 points) Determine true or false, with justification.
   (a) Let $\phi : \mathbb{Z}[x] \to \mathbb{R}$ denote the ring homomorphism defined by $\phi(x) = 1 + \sqrt{2}$. Then the kernel of ϕ is a principal ideal of $\mathbb{Z}[x]$.
   (b) The polynomial $x^4 + 3x + 6$ is irreducible over the field $\mathbb{Q}(\sqrt[3]{5})$.
   (c) Let F be a finite field and let $f \in F[x]$ be an irreducible polynomial of degree $n > 0$. Let K be a splitting field of f over F. Then f has n distinct roots in K.

2. (15 points) Determine if the following polynomials are irreducible in the respective rings.
   (a) $x^5 + 5x + 5 \in \mathbb{F}_2[x]$.
   (b) $x^4 + 1 \in \mathbb{Z}[x]$.
   (c) $x^{14} + 8x^{13} + 3 \in \mathbb{Q}[x]$.

3. (10 points) Suppose that $\alpha \in \mathbb{C}$ satisfies $\alpha^{15} - 42\alpha^{13} + 12\alpha^7 + 9\alpha^4 + 21\alpha - 24 = 0$. Show that $\mathbb{Q}(\alpha) = \mathbb{Q}(\alpha^2)$.

4. (10 points) Let K be a finite field of *odd* characteristic. Let q be the order of K.
   (a) Find the cardinality of the set $A := \{\alpha^2 \mid \alpha \in K\}$.
   (b) Find the cardinality of the set $B := \{\alpha^2 + \beta^2 \mid \alpha, \beta \in K\}$.

Do *any* two of the following three problems. You must declare which two should be graded in the appropriate box at the top of Page 1 of your solutions.

5. (15 points) Let K/F be a finite extension of fields and suppose that the characteristic of F is 0. Show that the set
$$\{L \mid L \text{ is a field such that } K \supset L \supset F\}$$
is finite.

6. (15 points) Let K/F be an algebraic extension of fields with the characteristic of F being 0. Suppose that every non-constant polynomial in $F[x]$ has a root in K. Show that K is algebraically closed. (That is, show that every non-constant polynomial in $K[x]$ has a root in K.)

7. (15 points) Let p be an arbitrary prime number and let $f = x^4 + 1 \in \mathbb{F}_p[x]$. Let K denote a splitting field of f over $\mathbb{F}_p$. Show that $[K : \mathbb{F}_p] \leq 2$.