ALGEBRA II MIDSEMESTER

(1) State whether true or false.
   (a) Every group of order 6 is abelian.
   (b) Every group of order 9 is abelian.
   (c) If $|G| = 18$ and $H < G$ such that $|H| = 9$ then H is normal in G.
   (d) Let G act on a set S and let $H = \{g \in G | g.s = s \forall s \in S\}$. Then H is normal in G.
   (e) The direct product of two cyclic groups is cyclic.
   (12 marks)

(2) Let $f : \mathbb{N} \to \mathbb{N}$ be defined as $f(n)$ is the number of groups of order n upto isomorphism. Is f a strictly increasing function? (5 marks)

(3) Define action of a group G on a set X. Define notions of orbits and stabilizers. Show that a group action of G on X gives a homomorphism $\phi : G \longrightarrow S(X)$. Describe the Kernel of ϕ. (Marks 5)

(4) Recall the school definitions of ${}^nP_r$ and ${}^nC_r$, namely the number of permutations of n things taken r at a time and the number of combinations of n things taken r. Show the relation between these two number using group theory. (8 marks)

(5) Let H be a subgroup of G. Define the centraliser of H as follows:
   $C_G(H) = \{g \in G | g.h.g^{-1} = h \forall h \in H\}$.
   Show that $C_G(H)$ is a normal subgroup of the normaliser $N_G(H)$ of H in G. State and prove Lagrange's theorem. (10 marks)

(6) Let G have a subgroup H of index n. Then there exists a normal subgroup $K \subset H$ such that the index of K in G divides $n!$. (10 marks)

(7) Let H be a subgroup of G and let G act on G/H as

$$g \cdot aH := gaH$$

   Show that the elements of G/H whose stabilizer contains H are the images of the elements of the normalizer $N(H)$ of H in G, under the natural quotient map $G \to G/H$. (10 marks)

(8) Let G be a group and A an abelian normal subgroup. Give an action of G/A on A and get in this manner a homomorphism of G/A into $Aut(A)$, where $Aut(A)$ is the group of automorphisms of the group A and is not just $S(A)$ the group of permutations of the set A. (10 marks)

~~e, a, ab, ba,~~

Date: 22 February 2023.