ALGEBRA II FINAL

(1) If A, B, C are subgroups of G, and $A \subset C$, then $(A.B) \cap C = A(B \cap C)$. (Marks 5)
(2) In the permutation group S_n give examples of an odd permutation of even order, and even permutations of even order. (5 marks)
(3) Describe groups of order 10. (marks 10)
(4) Let $i : G \hookrightarrow S(X)$ and let G be **abelian** such that G acts **transitively** on X via i. Show that $|G| = |X|$. (marks 5)
(5) Let G be a finite abelian group of order m. Let n be an integer coprime to m. Prove that every element $g \in G$ can be expressed as $g = x^n$ for some $x \in G$. (5 marks)
(6) Let G be a p-group and suppose that A is a normal subgroup of G such that $|A| = p$. Show that $A \subset Z(G)$. (marks 8)
(7) Let G be a finite group such that for every pair of elements $g, h \neq 1$, there exists an automorphism ϕ such that $\phi(g) = h$. Show that all elements have the same order which is a prime. Show that automorphisms send the center to itself. Hence show further that G is in fact abelian. (10 marks)
(8) Let $|G| = p^a.q^b$, where $p < q$ are primes. Suppose that the q-Sylow subgroup is normal in G. Show that there is a subgroup $H \subset G$ of index p which is normal. (12 marks)
(9) Let S and T be symmetric linear transformations on a finite dimensional real vector space V, such that $ST = TS$. Show that S and T have a common eigenvector in V. (5 marks)
(10) Show that if $A \in SO(n)$, with n-odd, then 1 is an eigenvalue of A. Conclude that every matrix in $SO(3)$ has an axis. (5 marks)

$|G| = m$
$(m, n) = 1$
$\{ab \mid a \in A, b \in B\}$
$a \in A, b \in B$
$(a, b) \longmapsto ab$
$a \in C, ab \in C$
$b = a^{-1}(ab) \in C$

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