Algebra 1 HW 5

Nirjhar Nath
nirjhar@cmi.ac.in
BMC202239

Problem 1:
Determine the primes p such that the matrix

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & -1 \\ -2 & 0 & 2 \end{bmatrix}$$

is invertible, when its entries are considered to be in $\mathbb{F}_p$.

Solution 1:
Here, we have

$$\begin{aligned} \det A &= 1 \det \begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix} - 2 \det \begin{bmatrix} 0 & -1 \\ -2 & 2 \end{bmatrix} + 0 \\ &= 6 - 2(-2) \\ &= 10 \end{aligned}$$

where 10 is the symbol that refers to $1 + 1 + \cdots 1$ (10 times).
We know that a square matrix over any field is invertible iff its determinant is non-zero (as a member of the field).
Now, $10 = 0$ in $\mathbb{F}_p$ iff $p \mid (10 - 0) = 10$ i.e., iff $p = 2$ or 5 in $\mathbb{F}_p$.
Thus, for any prime p in $\mathbb{F}_p$ except for $2 = 1 + 1$ and $5 = 1 + 1 + 1 + 1 + 1$ in $\mathbb{F}_p$, the given matrix is invertible. ■

Problem 2:
Solve completely the systems of linear equations $AX = 0$ and $AX = B$, where

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

(a) in $\mathbb{Q}$, **(b)** in $\mathbb{F}_2$, **(c)** in $\mathbb{F}_3$, **(d)** in $\mathbb{F}_7$.

Solution 2:
(a) Consider the augmented matrix

$$[A|0] = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right]$$

Using $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get the matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -2 & -1 & 0 \end{array}\right]$$

Using $R_3 \to R_3 - 2R_2$, we get the matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

Using $R_3 \to -R_3$ and $R_2 \to -R_2$, we get the matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Using $R_2 \to R_2 + R_3$, we get the matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Using $R_1 \to R_1 - R_2$, we get the matrix (with A in RREF)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Thus, the solution to the system $AX = 0$ with A in $\mathbb{Q}$ is $X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Now consider the augmented matrix

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{array}\right]$$

Using same row operations as above, we get the matrix (with A in RREF)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Thus, the solution to the system $AX = B$ with A, B in $\mathbb{Q}$ is $X = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$.

(b) In $\mathbb{F}_2$, the operations $+$ and $\cdot$ are defined by

+	0	1
0	0	1
1	1	0

and

$\cdot$	0	1
0	0	0
1	0	1

Consider the augmented matrix

$$[A|0] = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right]$$

Using $R_2 \to R_2 + R_1$ and $R_3 \to R_3 + R_1$, we get the matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

Using $R_2 \to R_2 + R_3$, we get the matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

Using $R_1 \to R_1 + R_2$ and replacing -1 by 1 in $\mathbb{F}_2$, we get the matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Thus, the solution to the system $AX = 0$ with A in $\mathbb{F}_2$ is $X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Now consider the augmented matrix

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{array}\right]$$

Using same row operations as above, we get the matrix (with A in RREF)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Thus, the solution to the system $AX = B$ with A, B in $\mathbb{F}_2$ is $X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

(c) Since all the row operations used in **(a)** of the form $R_i \to R_i + \lambda R_j$, $i \neq j$

are such that λ is an integer. So, for all such row operations, we can replace λ by $\lambda \pmod 3$ in $\mathbb{F}_3$. So, we can replace all the entries in $[A|B]$ (with A in RREF) with their values mod 3, to get $[A|B]$ (with A in RREF) in $\mathbb{F}_3$.
So the augmented matrix $[A|0]$, with A in RREF is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Thus, the solution to the system $AX = 0$ with A in $\mathbb{F}_3$ is $X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Also, the augmented matrix $[A|B]$, with A in RREF is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Thus, the solution to the system $AX = B$ with A, B in $\mathbb{F}_2$ is $X = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

(d) Since all the row operations used in **(a)** of the form $R_i \to R_i + \lambda R_j$, $i \neq j$ are such that λ is an integer. So, for all such row operations, we can replace λ by $\lambda \pmod 7$ in $\mathbb{F}_7$. So, we can replace all the entries in $[A|B]$ (with A in RREF) with their values mod 7, to get $[A|B]$ (with A in RREF) in $\mathbb{F}_7$.
So the augmented matrix $[A|0]$, with A in RREF is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Thus, the solution to the system $AX = 0$ with A in $\mathbb{F}_7$ is $X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Also, the augmented matrix $[A|B]$, with A in RREF is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Thus, the solution to the system $AX = B$ with A, B in $\mathbb{F}_2$ is $X = \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix}$. $\blacksquare$

Problem 3:
Let $\mathbb{F}_p$ be a prime field, and let $V = \mathbb{F}_p^2$. Prove:

(a) The number of bases of V is equal to the order of the general linear group $GL_2(\mathbb{F}_p)$.

(b) The order of the general linear group $GL_2(\mathbb{F}_p)$ is $p(p+1)(p-1)^2$, and the order of the special linear group $SL_2(\mathbb{F}_p)$ is $p(p+1)(p-1)$.

Solution 3:

(a) Consider the map ϕ from the ordered bases of $V = \mathbb{F}_p^2$ to $GL_2(\mathbb{F}_p)$ given by $\phi(v_1, v_2) = [v_1, v_2]$. Clearly, the map is injective. Also, the map is surjective, since the columns of an invertible matrix form an ordered basis. Thus, ϕ is a bijection and hence the number of bases of V is equal to the order of the general linear group $GL_2(\mathbb{F}_p)$.

(b) Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix in $GL_2(\mathbb{F}_p)$. So, M is invertible iff $ad \neq bc$. If a, b, c, d are all non-zero, we can fix a, b, c arbitrarily and d can be anything except $a^{-1}bc$. This gives us $(p-1)^3(p-2)$ matrices. If exactly one of the entries is 0, then the other three entries can be anything non-zero, for a total of $4(p-1)^3$ matrices. Now, if exactly two entries are 0, then these entries must be opposite to each other for the matrix to be invertible, and the other two entries can be anything non-zero, for a total of $2(p-1)^2$ matrices. So the order is

$$\begin{aligned} &(p-1)^3(p-2) + 4(p-1)^3 + 2(p-1)^2 \\ =&(p-1)^2[(p-1)(p-2) + 4(p-1) + 2] \\ =&(p-1)^2[(p-1)(p+2) + 2] \\ =&(p-1)^2(p^2+p) \\ =&p(p+1)(p-1)^2 \end{aligned}$$

Thus, the order of the general linear group $GL_2(\mathbb{F}_p)$ is $p(p+1)(p-1)^2$. Now, construct a map f such that if we multiply the first row by some non-zero k in $\mathbb{F}_p$, any matrix with determinant 1 is mapped to a matrix with determinant k and any matrix with determinant k is mapped to a matrix with determinant 1 (all matrices in $GL_2(\mathbb{F}_p)$). Thus, we have produced a bijection f between matrices with determinant 1 and matrices with determinant k. Now there are $(p-1)$ possibilities for k. It follows that the order of the special linear group $SL_2(\mathbb{F}_p)$ is

$$|SL_2(\mathbb{F}_p)| = \frac{1}{p-1}|GL_2(\mathbb{F}_p)| = \frac{1}{p-1}\{p(p+1)(p-1)^2\} = p(p+1)(p-1)$$

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Problem 4:
How many subspaces of each dimension are there in $\mathbb{F}_p^3$?

Solution 4:
A k-dimensional subspace of a vector space $\mathbb{F}_p^3$ is specified by giving k linearly independent vectors $\{v_1, v_2, \ldots, v_k\}$ in $\mathbb{F}_p^3$ with $k = 1, 2, 3$. Firstly, v_1 can be taken to be any non-zero vector in V. Therefore there are $p^3 - 1$ choices for v_1. Given v_1, v_2 can be chosen to be any vector which is not in the subspace spanned by v_1. Since this subspace has p elements, there are $p^3 - p$ choices for v_2. Similarly, given v_1, v_2, there are $p^3 - p^2$ choices for v_3. Therefore, the number of sets of k linearly independent vectors in $\mathbb{F}_p^3$ is $\prod_{i=1}^{k}(p^3 - p^{i-1})$.
For $k = 3$, we see that each k-dimensional subspace of $\mathbb{F}_p^3$ has $\prod_{i=1}^{k}(p^k - p^{i-1})$ bases. Thus the number of k-dimensional subspaces of $\mathbb{F}_p^3$ is

$$\frac{\prod_{i=1}^{k}(p^3 - p^{i-1})}{\prod_{i=1}^{k}(p^k - p^{i-1})}$$

for $k = 1, 2, 3$. ■

Problem 5:
Consider the determinant function det : $F^{2\times 2} \to F$, where $F = \mathbb{F}_p$ is the prime field of order p and $F^{2\times 2}$ is the space of 2×2 matrices. Show that this map is surjective, that all non-zero values of the determinant are taken on the same number of times, but that there are more matrices with determinant 0 than with determinant 1.

Solution 5:
In order to prove that det : $F^{2\times 2} \to F$ (where $F = \mathbb{F}_p$) is surjective, observe that for any $k \in \mathbb{F}_p$,

$$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} = k$$

Thus, det is surjective.
There are exactly p^4 matrices with each entry from $\mathbb{F}_p$.
From Solution 3, the number of matrices with non-zero determinant is the order of the general linear group given by $GL_2(\mathbb{F}_p)$ is

$$|GL_2(\mathbb{F}_p)| = p(p+1)(p-1)^2$$

Therefore, the number of matrices with determinant 0 is

$$p^4 - GL_2(\mathbb{F}_p) = p^4 - (p^4 - p^3 - p^2 + p) = p^3 + p^2 - p$$

Also from Solution 3, the number of matrices with determinant 1 is the order of the special linear group given by $SL_2(\mathbb{F}_p)$ is

$$|SL_2(\mathbb{F}_p)| = p(p+1)(p-1) = p^3 - p$$

Clearly for $p^2 > 0$, we have

$$p^3 + p^2 - p > p^3 - p$$

i.e., there are more matrices with determinant 0 than with determinant 1.■

Problem 6:
A 2×2 matrix A has an eigenvector $v_1 = (1,1)^t$ with eigenvalue 2 and also an eigenvector $v_2 = (1,2)^t$ with eigenvalue 3. Determine A.

Solution 6:
Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
Since A has an eigenvector $v_1 = (1,1)^t$ with eigenvalue 2, so

$$\begin{aligned} & Av_1 = 2v_1 \\ \Longrightarrow & \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \Longrightarrow & \begin{bmatrix} a+b \\ c+d \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \end{aligned}$$

Since A also has an eigenvector $v_2 = (1,2)^t$ with eigenvalue 3, so

$$\begin{aligned} & Av_1 = 2v_1 \\ \Longrightarrow & \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ \Longrightarrow & \begin{bmatrix} a+2b \\ c+2d \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \end{aligned}$$

Thus, we have the following two system of equations

$$\begin{cases} a + b = 2 \\ a + 2b = 3 \end{cases} \quad \text{and} \quad \begin{cases} c + d = 2 \\ c + 2d = 6 \end{cases}$$

Solving, we get $a = 1, b = 1, c = -2, d = 4$. Thus, the matrix is

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$$

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Problem 7:
Compute the characteristic polynomial and the complex eigenvalues and eigenvectors of

$$\begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix}$$

Solution 7:
Let $A = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix}$. Then the characteristic polynomial of A is given by

$$p(\lambda) = \det(\lambda I_2 - A) = \det \begin{bmatrix} \lambda + 2 & -2 \\ 2 & \lambda - 3 \end{bmatrix} = \lambda^2 - \lambda - 2$$

Hence the complex eigenvalues of A are the roots of

$$p(\lambda) = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0$$

which are $2, -1$.
Let us now find the eigenvectors corresponding to the eigenvalue 2. We seek a non-zero column vector $\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$ such that

$$\begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = 2 \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

i.e.,

$$\begin{bmatrix} -4\lambda_1 + 2\lambda_2 \\ -2\lambda_1 + \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus, any column vector $\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} k \\ 2k \end{bmatrix}$ for non-zero k is an eigenvector corresponding to 2.
Now we find the eigenvectors corresponding to the eigenvalue -1. We seek a non-zero column vector $\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$ such that

$$\begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = -1 \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

i.e.,

$$\begin{bmatrix} -\lambda_1 + 2\lambda_2 \\ -2\lambda_1 + 4\lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus, any column vector $\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 2k \\ k \end{bmatrix}$ for non-zero k is an eigenvector corresponding to -1. ■

Problem 8:
The characteristic polynomial of the matrix below is $t^3 - 4t - 1$. Determine the missing entries.

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 1 & * & * \end{bmatrix}$$

Solution 8:
Taking x and y as the missing entries, consider the matrix as

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 1 & x & y \end{bmatrix}$$

The characteristic polynomial of A is given by

$$\begin{aligned}
&\det(tI_3 - A) \\
=&\det \begin{bmatrix} t & -1 & -2 \\ -1 & t-1 & 0 \\ -1 & -x & t-y \end{bmatrix} \\
=&t \det \begin{bmatrix} t-1 & 0 \\ -x & t-y \end{bmatrix} + 1 \det \begin{bmatrix} -1 & 0 \\ -1 & t-y \end{bmatrix} - 2 \begin{bmatrix} -1 & t-1 \\ -1 & -x \end{bmatrix} \\
=&t(t-1)(t-y) - t(t-y) - 2(x+t-1) \\
=&t^3 - (y+1)t^2 + (y-3)t + (-2x+y+2)
\end{aligned}$$

Now, we should have

$$t^3 - (y+1)t^2 + (y-3)t + (-2x+y+2) = t^3 - 4t - 1$$

So, $y + 1 = 0, y - 3 = -4$ and $-2x + y + 2 = -1$. So, $x = 1$ and $y = -1$ satisfy the equations. Thus, the matrix is

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

■