Algebra 1 HW 4

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Problem 1:
(a) Prove that the space $\mathbb{R}^{n\times n}$ of all $n \times n$ real matrices is the direct sum of the space of symmetric matrices ($A^t = A$) and the space of skew-symmetric matrices ($A^t = -A$).
(b) The trace of a square matrix is the sum of its diagonal entries. Let W_1 be the space of $n \times n$ matrices whose trace is zero. Find a subspace W_2 so that $\mathbb{R}^{n\times n} = W_1 \oplus W_2$.
Solution 1:
(a) Let V_1 and V_2 be the spaces of symmetric and skew-symmetric matrices respectively of $\mathbb{R}^{n\times n}$.
Suppose

$$V_1 \oplus V_2 = \mathbb{R}^{n\times n}$$

So, $V_1 + V_2 = \mathbb{R}^{n\times n}$ and $V_1 \cap V_2 = 0$.
We have,

$$A = \frac{1}{2}(A + A^t) + \frac{1}{2}(A - A^t)$$

where

$$\left\{\frac{1}{2}(A + A^t)\right\}^t = \frac{1}{2}(A^t + A)$$

and

$$\left\{\frac{1}{2}(A - A^t)\right\}^t = \frac{1}{2}(A^t - A) = -\frac{1}{2}(A - A^t)$$

Therefore, $\frac{1}{2}(A + A^t) \in V_1$ and $\frac{1}{2}(A - A^t) \in V_2$.
Thus, assuming $M_1 = \frac{1}{2}(A + A^t)$ and $M_2 = \frac{1}{2}(A - A^t)$, we have

$$M = M_1 + M_2 \ \text{ for } \ M_1 \in V_1, M_2 \in V_2$$

Therefore,

$$V_1 + V_2 = \mathbb{R}^{n\times n}$$

Now, if $A = A^t$ and $A = -A^t$, adding we have $A = 0$, the null matrix.
Therefore,

$$V_1 \cap V_2 = \{0\}$$

Hence,

$$V_1 \oplus V_2 = \mathbb{R}^{n\times n}$$

(b) Let W_1 be the space of $n \times n$ matrices whose trace is zero. Let W_2 be the subspace of $\mathbb{R}^{n\times n}$ and

$$W_2 = k(e_{11}) + 0(e_{12}) + \cdots + 0(e_{nn}) = \begin{bmatrix} k & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Suppose M is a matrix of $\mathbb{R}^{n\times n}$ with trace p. For any such M we can construct W' such that $M = W + W'$ with $W \in W_1$.
Construct

$$W = M - p \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Therefore,

$$\text{tr }(W) = \text{tr }(M) - p = p - p = 0$$

Thus, $W \in W_1$ with $\text{tr }(W) = 0$.
Thus,

$$W_1 + W_2 = \mathbb{R}^{n\times n}$$

.
Now, if $W \in W_1$, then $\text{tr }(W) = 0$ and if $V \in W_2$, then $\text{tr }(V) = p$.
Thus, if A is any arbitrary matrix such that $A \in W_1 \cup W_2$, then $A = W + V$.
Therefore,

$$\text{tr }(A) = \text{tr }(W + V) = \text{tr }(W) + \text{tr }(V)$$

But, $\text{tr }(A) = 0$ since A lies in V_1.
Therefore,

$$\text{tr }(V) = \text{tr }(A) - \text{tr }(W) = 0 - 0 = 0$$

So,

$$V = \begin{bmatrix} p & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} = 0$$

which is the null matrix. Thus,

$$W_1 \cap W_2 = \{0\}$$

Hence,

$$\mathbb{R}^{n\times n} = W_1 \oplus W_2$$

$\square$

Problem 2:
Let E be the set of vectors $(e_1, e_2, \dots)$ in $\mathbb{R}^\infty$ and let $w = (1, 1, 1, \dots)$. Describe the span of the set $(w, e_1, e_2, \dots)$. Also find a vector that is not in the span of the set $(w, e_1, e_2, \dots)$.
Solution 2:
The span of the infinite set $(w, e_1, e_2, \dots)$ is defined to be the set of the vectors v that are combinations of finitely many elements of $(w, e_1, e_2, \dots)$, i.e.,

$$v = c_1 x_1 + c_2 x_2 + \cdots c_n v_n$$

where x_i's are in $(w, e_1, e_2, \dots)$. The set also does not span because the set $(1, 2, 3, \dots)$ cannot be written as a finite combination of elements of the set $(w, e_1, e_2, \dots)$. Thus, $(1, 2, 3, \dots)$ is a vector that is not in the span of the set $(w, e_1, e_2, \dots)$. $\square$
Problem 3:
Compute the determinant of the following $n \times n$ matrix using induction on n:

$$\begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & & & & \\ & -1 & 2 & & & \\ & & & \ddots & & \\ & & & & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}$$

Solution 3: Let

$$A_n = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & & & & \\ & -1 & 2 & & & \\ & & & \ddots & & \\ & & & & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}_n$$

be a $n \times n$ matrix. Thus, we can expand $\det A_n$ as

$$2 \det \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & & & & \\ & -1 & 2 & & & \\ & & & \ddots & & \\ & & & & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}_{n-1} - (-1) \det \begin{bmatrix} -1 & -1 & & & & \\ 0 & 2 & -1 & & & \\ & -1 & 2 & & & \\ & & & \ddots & & \\ & & & & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}_{n-1}$$

Thus,

$$\det A_n = 2 \det A_{n-1} - \det A_{n-2}$$

We claim that $\det A_n = n + 1$.
Clearly, $\det A_1 = \det[2] = 2$,
$\det A_2 = \det \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 4 - 1 = 3$.
We use strong induction to prove our claim.
Suppose $\det A_i = i + 1$ for all $1 \leq i \leq k$ for some $k \in \mathbb{N}$.
Thus,

$$\det A_{k+1} = 2 \det A_k - \det A_{k-1} = 2(k+1) - k = k + 2$$

Thus, by induction, $\det A_n = n + 1$ for all n. $\square$

Problem 4:
Prove that $\det \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} = (\det A)(\det D)$, if A and D are square blocks.
Solution 4:
If D is not invertible, the set of row vectors of D is not linearly independent i.e., the set of row vectors $(0 \quad D)$ is also not linearly independent. So, $\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$ cannot have all independent rows and hence not invertible.
Also, since D is square and not invertible, it must be the case that $\det D = 0$. So,

$$(\det A)(\det D) = 0 = \det \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$$

If D is invertible, then we have

$$\begin{bmatrix} I & 0 \\ 0 & D^{-1} \end{bmatrix} \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & I \end{bmatrix}$$

$$\Rightarrow \det \begin{bmatrix} I & 0 \\ 0 & D^{-1} \end{bmatrix} \det \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} = \det \begin{bmatrix} A & B \\ 0 & I \end{bmatrix}$$

So we get the identity

$$\det D^{-1} \det \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} = \det A$$

Using $\det A^{-1} = \dfrac{1}{\det A}$, we have

$$\det \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} = (\det A)(\det D)$$

□

Problem 5:
Compute the determinant of the following matrix by expansion on the bottom row:

$$\begin{bmatrix} a & b & c \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Solution 5:
Expanding on the bottom row, we have

$$\det \begin{bmatrix} a & b & c \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = 1 \det \begin{bmatrix} b & c \\ 0 & 1 \end{bmatrix} - 1 \det \begin{bmatrix} a & c \\ 1 & 1 \end{bmatrix} + 1 \det \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}$$

$= b - (a - c) + (-b) = c - a$ □

Problem 6:
Let A be an $n \times n$ matrix with integer entries a_{ij}. Prove that A is invertible, and that its inverse A^{-1} has integer entries, if and only if $\det A = \pm 1$.
Solution 6:
First we prove that A is invertible.
Suppose that $\det A \neq 0$. Let e_i be the standard basis vector. Then the equations $Ax = e_i$ has a unique solution x_i for $i = 1, 2, \ldots, n$.
Construct $B = (x_1, x_2, \ldots, x_n)$. Then we have $AB = (e_1, e_2, \ldots, e_n) = I_n$.
So, B is the right inverse of A and hence A is the right inverse of B and thus, A is invertible with $A^{-1} = B$.
Now suppose A^{-1} has integer entries, then we have

$$\det A^{-1} \det A = \det(A^{-1}A) = \det I = 1$$

or,

$$\det A^{-1} = \frac{1}{\det A} \in \mathbb{Z}$$

which is only possible when $\det A = \pm 1$.
To prove the other direction, we proceed as follows.
We have,

$$A^{-1} = \frac{1}{\det A}(\text{adj } A)$$

Now if $\det A = \pm 1$, we have $A^{-1} = \pm \text{adj } A$. Now, since adj A is the matrix of co-factors of A, and since A has all integer entries, so adj A will have all integer entries. Hence A^{-1} has all integer entries. □

Problem 7: (*Vandermonde determinant*)
(a) Prove that $\det \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix} = (a-b)(b-c)(c-a)$.
(b) Prove an analogous formula for $n \times n$ matrices, using appropriate row operations to clear out the first column.
(c) Use the Vandermonde determinant to prove that there is a unique polynomial $p(t)$ of degree n that takes arbitrary prescribed values at $n+1$ points to $t_0, \ldots, t_n$.
Solution 7:
(a)

$$\det \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix} = 1(bc^2 - b^2c) - 1(c^2a - ca^2) + 1(ab^2 - a^2b)$$

$$= ab^2 + bc^2 + ca^2 - a^2b - b^2c - ca^2 = (a-b)(b-c)(c-a)$$

(b) Let

$$A_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \\ a_1^2 & a_2^2 & a_3^2 & \cdots & a_n^2 \\ \vdots & \vdots & & & \vdots \\ a_1^{n-1} & a_2^{n-1} & a_3^{n-1} & \cdots & a_n^{n-1} \end{bmatrix}$$

Therefore,

$$\det A_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \\ a_1^2 & a_2^2 & a_3^2 & \cdots & a_n^2 \\ \vdots & \vdots & & & \vdots \\ a_1^{n-1} & a_2^{n-1} & a_3^{n-1} & \cdots & a_n^{n-1} \end{bmatrix} = \det \begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ 1 & a_3 & a_3^2 & \cdots & a_3^{n-1} \\ \vdots & \vdots & & & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{bmatrix}$$

Using row operations $R_i \to R_i - R_1$ for all $2 \leq i \leq n$, we get

$$\det A_n = \det \begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 - a_1 & a_2^2 - a_1^2 & \cdots & a_2^{n-1} - a_1^{n-1} \\ 1 & a_3 - a_1 & a_3^2 - a_1^2 & \cdots & a_3^{n-1} - a_1^{n-1} \\ \vdots & \vdots & & & \vdots \\ 1 & a_n - a_1 & a_n^2 - a_1^2 & \cdots & a_n^{n-1} - a_1^{n-1} \end{bmatrix}$$

Without changing the value of $\det A_n$, we perform the column operations $C_i \to C_i - C_{i-1}$ for $2 \leq i \leq n$ to get

$$\det A_n = \det \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 - a_1 & (a_2 - a_1)a_2 & \cdots & (a_2 - a_1)a_2^{n-2} \\ 0 & a_3 - a_1 & (a_3 - a_1)a_3 & \cdots & (a_3 - a_1)a_3^{n-2} \\ \vdots & \vdots & & & \vdots \\ 0 & a_n - a_1 & (a_n - a_1)a_n & \cdots & (a_n - a_1)a_n^{n-2} \end{bmatrix}$$

$$= \prod_{i=2}^{n} (a_i - a_1) \det \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & a_2 & \cdots & a_2^{n-2} \\ 0 & 1 & a_3 & \cdots & a_3^{n-2} \\ \vdots & \vdots & & & \vdots \\ 0 & 1 & a_n & \cdots & a_n^{n-2} \end{bmatrix}$$

$$= \prod_{i=2}^{n} (a_i - a_1) \det \begin{bmatrix} 1 & a_2 & \cdots & a_2^{n-2} \\ 1 & a_3 & \cdots & a_3^{n-2} \\ \vdots & \vdots & & \vdots \\ 1 & a_n & \cdots & a_n^{n-2} \end{bmatrix}$$

Thus, we get the recursive formula

$$\det A_n = \prod_{i=2}^{n}(a_i - a_1)\det A_{n-1} \tag{1}$$

We have

$$\det A_2 = \det\begin{bmatrix} 1 & 1 \\ a_{n-1} & a_n \end{bmatrix} = \det\begin{bmatrix} 1 & a_{n-1} \\ 1 & a_n \end{bmatrix} = a_n - a_{n-1}$$

Using this in (1), we have

$$\det A_n = \prod_{1\le j<i\le n}(a_i - a_j)$$

(c) Let $p(t)$ be a polynomial such that $p(t_i) = a_i$ for all $0 \le i \le n$.
So,

$$p(t) = \sum_{1\le i<j\le n} a_i \frac{(t-t_j)}{(t_i - t_j)}$$

Suppose $p(t)$ and $f(t)$ are two unique polynomials of degree n such that they have same values for $t = t_0, t_1, \ldots, t_n$.
Then $\deg(p(t) - f(t)) \le n$ and $p(t) - f(t)$ for all t.
Let $q(t)$ be a polynomial such that $q(t_i) = 0$ for all $0 \le i \le n$. Then,

$$q(t) = p(t) - f(t)$$

Now suppose

$$q(t) = \sum_{i=0}^{n} \lambda_i t^i$$

where λ_i's are not all zero.
Now,

$$det\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ t_1 & t_2 & t_3 & \cdots & t_n \\ t_1^2 & t_2^2 & t_3^2 & \cdots & t_n^2 \\ \vdots & \vdots & & & \vdots \\ t_1^{n-1} & t_2^{n-1} & t_3^{n-1} & \cdots & t_n^{n-1} \end{bmatrix} = \prod_{0\le i<j\le n}(t_i - t_j) \neq 0 \tag{2}$$

But if $q(t_i) = 0$ for all $0 \le i \le n$, then

$$\sum_{0\le i\le n} \lambda_i R_i = 0$$

where R_i is the i^{th} row.
This implies,

$$R_k = \sum_{0 \le i < j < n} \frac{\lambda_i}{\lambda_j} R_i = 0$$

a contradiction to equation (2).
Thus, $p(t) = f(t)$. $\square$

Problem 8:
Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 10 & 20 & 30 & 40 & 50 \\ 3 & 6 & 7 & 18 & 14 \end{bmatrix}$. Consider the map $T(x) = Ax$ from $\mathbb{R}^5$ to $\mathbb{R}^3$. Find an explicit basis of $\mathbb{R}^5$ and an explicit basis of $\mathbb{R}^3$ such that with respect to these bases the matrix of T has matrix $M =$ identity of suitable size in top left corner and zeros everywhere else.

Solution 8:
Given,

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 10 & 20 & 30 & 40 & 50 \\ 3 & 6 & 7 & 18 & 14 \end{bmatrix}$$

Using row operations $R_2 \to R_2 - 10R_1$ and $R_3 \to R_3 - 3R_1$, we get the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 6 & -1 \end{bmatrix}$$

Now by $R_2 \leftrightarrow R_3$, we get the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & -2 & 6 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Using row operation $R_2 \to -\frac{1}{2}R_2$, we get the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & -3 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Using column operations $C_i = C_i - iC_1$ for $i = 2, 3, 4, 5$, we get the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now using column operations $C_4 \to C_4 + 3C_3$ and $C_5 \to C_5 - \frac{1}{2}C_3$, we get the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now using $C_2 \leftrightarrow C_3$, we get the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We know that every row and column operations on a matrix are respectively left and right multiplication of the matrix by corresponding elementary matrices.

Therefore, A is left multiplied by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -10 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -10 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 0 & -\frac{1}{2} \\ -10 & 1 & 0 \end{bmatrix}$$

Also, A is right multiplied by

$$\begin{bmatrix} 1 & -2 & -3 & -4 & -5 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & -3 & -13 & -\frac{7}{2} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 & -2 & -13 & -\frac{7}{2} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus,

$$A' = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 0 & -\frac{1}{2} \\ -10 & 1 & 0 \end{bmatrix} A \begin{bmatrix} 1 & -3 & -2 & -13 & -\frac{7}{2} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

So,

$$A' = \begin{bmatrix} 1 & 0 & 0 \\ 10 & 0 & 1 \\ -3 & -2 & 0 \end{bmatrix}^{-1} A \begin{bmatrix} 1 & -3 & -2 & -13 & -\frac{7}{2} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = Q^{-1}AP$$

where

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 10 & 0 & 1 \\ -3 & -2 & 0 \end{bmatrix}$$

and

$$P = \begin{bmatrix} 1 & -3 & -2 & -13 & -\frac{7}{2} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We had old basis of $\mathbb{R}^3$ and $\mathbb{R}^5$ as the standard basis.
Thus, new basis of $\mathbb{R}^3$ is $(e_1, e_2, e_3)Q = ((1, 10, -3)^t, (0, 0, -2)^t, (0, 1, 0)^t)$ and new basis of $\mathbb{R}^5$ is $(e_1, e_2, e_3, e_4, e_5)P = ((1, 0, 0, 0, 0)^t, (-3, 0, 1, 0, 0)^t, (-2, 1, 0, 0, 0)^t, (-13, 0, 3, 1, 0)^t, (-\frac{7}{2}, 0, -\frac{1}{2}, 0, 1)^t)$. □