Algebra 1 HW 3

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Problem 1:

(a) Let $\mathbf{B} = ((1,2,0)^t, (2,1,2)^t, (3,1,1)^t)$ and $\mathbf{B}' = ((0,1,0)^t, (1,0,1)^t, (2,1,0)^t)$. Determine the basechange matrix P from $\mathbf{B}$ to $\mathbf{B}'$.

(b) Determine the basechange matrix in $\mathbb{R}^2$, when the old basis is the standard basis $\mathbf{E} = (e_1, e_2)$ and the new basis is $\mathbf{B} = (e_1 + e_2, e_1 - e_2)$.

(c) Determine the basechange matrix in $\mathbb{R}^n$, when the old basis is the standard basis $\mathbf{E}$ and the new basis is $\mathbf{B} = (e_n, e_{n-1}, \ldots, e_1)$.

Solution 1:

(a) Since P is the basechange matrix from $\mathbf{B}$ to $\mathbf{B}'$, so $\mathbf{B}' = \mathbf{B}P$.

Suppose for some x_1, y_1, z_1,

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + y_1 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + z_1 \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

i.e.,

$$\begin{cases} x_1 + 2y_1 + 3z_1 = 0 \\ 2x_1 + y_1 + z_1 = 1 \\ 2y_1 + z_1 = 0 \end{cases}$$

So we have $x_1 = \frac{4}{7}, y_1 = \frac{1}{7}, z_1 = -\frac{2}{7}$.

Suppose for some x_2, y_2, z_2,

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + y_2 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + z_2 \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

i.e.,

$$\begin{cases} x_2 + 2y_2 + 3z_2 = 1 \\ 2x_2 + y_2 + z_2 = 0 \\ 2y_2 + z_2 = 1 \end{cases}$$

So we have $x_2 = -\frac{2}{7}, y_2 = \frac{3}{7}, z_2 = \frac{1}{7}$.

Suppose for some x_3, y_3, z_3,

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + y_3 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + z_3 \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

i.e.,

$$\begin{cases} x_3 + 2y_3 + 3z_3 = 2 \\ 2x_3 + y_3 + z_3 = 1 \\ 2y_3 + z_3 = 0 \end{cases}$$

So we have $x_3 = \frac{2}{7}, y_3 = -\frac{3}{7}, z_3 = \frac{6}{7}$.
Thus, the basechange matrix from $\mathbf{B}$ to $\mathbf{B}'$ is

$$P = \begin{bmatrix} \frac{4}{7} & -\frac{2}{7} & \frac{2}{7} \\ \frac{1}{7} & \frac{3}{7} & -\frac{3}{7} \\ -\frac{2}{7} & \frac{1}{7} & \frac{6}{7} \end{bmatrix}$$

(b) We have, $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. So,

$$e_1 + e_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } e_1 - e_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Suppose $\mathbf{B} = \mathbf{E}P$, where P is the basechange matrix in $\mathbb{R}^2$, i.e.,

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} P = IP$$

So, $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.
(c) Suppose the basechange matrix in $\mathbb{R}^n$ is P, when the old basis is the standard basis $\mathbf{E}$ and the new basis is $\mathbf{B} = (e_n, e_{n-1}, \ldots, e_1)$, i.e., $\mathbf{B}P = \mathbf{E}$. But the matrix formed by the standard basis vectors is the $n \times n$ identity matrix I. So, $P = \mathbf{B}^{-1}$.

Problem 2:
Prove that every $m \times n$ matrix A of rank 1 has the form $A = XY^t$, where X, Y are m- and n-dimensional column vectors. How uniquely determined are these vectors?

Solution 2:
If A is a $m \times n$ matrix A of rank 1, then A has only one independent column and every other column is a multiple of it. Let $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \in \mathbb{R}^m$, $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n$ and

$$A = XY^t = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} [y_1 \ \ y_2 \ \ \cdots \ \ y_n] = [y_1 X \ \ y_2 X \ \ \cdots \ \ y_n X]$$

So every column of A is a multiple of the column vector X, i.e., the matrix A has rank 1.

Problem 3:

Let A and B be 2×2 matrices. Determine the matrix of the operator $T : M \rightsquigarrow AMB$ on the space $F^{2\times 2}$ of 2×2 matrices, with respect to the basis $(e_{11}, e_{12}, e_{21}, e_{22})$ of $F^{2\times 2}$.

Solution 3:

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$.

Therefore,

$$\begin{aligned} T(e_{11}) &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \\ &= \begin{bmatrix} a_{11} & 0 \\ a_{21} & 0 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} \end{bmatrix} \\ &= a_{11}b_{11}(e_{11}) + a_{11}b_{12}(e_{12}) + a_{21}b_{21}(e_{21}) + a_{21}b_{22}(e_{22}) \end{aligned}$$

Now,

$$\begin{aligned} T(e_{12}) &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \\ &= \begin{bmatrix} 0 & a_{11} \\ 0 & a_{21} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{21} & a_{11}b_{22} \\ a_{21}b_{21} & a_{21}b_{22} \end{bmatrix} \\ &= a_{11}b_{21}(e_{11}) + a_{11}b_{22}(e_{12}) + a_{21}b_{21}(e_{21}) + a_{21}b_{22}(e_{22}) \end{aligned}$$

Now,

$$\begin{aligned} T(e_{21}) &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \\ &= \begin{bmatrix} a_{12} & 0 \\ a_{22} & 0 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{12}b_{11} & a_{12}b_{12} \\ a_{22}b_{11} & a_{22}b_{12} \end{bmatrix} \\ &= a_{12}b_{11}(e_{11}) + a_{12}b_{12}(e_{12}) + a_{22}b_{11}(e_{21}) + a_{22}b_{12}(e_{22}) \end{aligned}$$

and

$$\begin{aligned} T(e_{22}) &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \\ &= \begin{bmatrix} 0 & a_{12} \\ 0 & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{12}b_{21} & a_{12}b_{22} \\ a_{22}b_{21} & a_{22}b_{22} \end{bmatrix} \\ &= a_{12}b_{21}(e_{11}) + a_{12}b_{22}(e_{12}) + a_{22}b_{21}(e_{21}) + a_{22}b_{22}(e_{22}) \end{aligned}$$

So we have

$$T = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{21} & a_{12}b_{11} & a_{12}b_{21} \\ a_{11}b_{12} & a_{11}b_{22} & a_{12}b_{12} & a_{12}b_{22} \\ a_{21}b_{21} & a_{21}b_{21} & a_{22}b_{11} & a_{22}b_{21} \\ a_{21}b_{12} & a_{21}b_{22} & a_{22}b_{12} & a_{22}b_{22} \end{bmatrix}$$

Problem 4:
Prove the below theorem using row and column operations:
Given an $m \times n$ matrix A, there are invertible matrices Q and P such that $A' = Q^{-1}AP$ has the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$.

Problem 5: Determine the dimensions of the kernel and the image of the linear operator T on the space $\mathbb{R}^n$ defined by

$$T(x_1, \ldots, x_n)^t = (x_1 + x_n, x_2 + x_n - 1, \ldots, x_n + x_1)^t$$

Solution 5:
The linear operator T on the space $\mathbb{R}^n$ is defined by

$$T(x_1, \ldots, x_n)^t = (x_1 + x_n, x_2 + x_n - 1, \ldots, x_n + x_1)^t$$

Therefore,

$$T(x_1e_1 + x_2e_2 + \cdots + x_ne_n) = x_1T(e_1) + x_2T(e_2) + \cdots + x_nT(e_n)$$

$$= (x_1 + x_n, x_2 + x_n - 1, \ldots, x_n + x_1)^t$$

Now, since e_i's are standard basis vectors, so

$$T(e_i) = e_i + e_{n-i+1}$$

for $i = 1, 2, \cdots, n$.
So,

$$T = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & 0 & \cdots & 1 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

In RREF, T' has a 1 in the last row at position $\frac{n}{2}$ or $\frac{n-1}{2}$ depending on whether n is even or odd.
For even n, dim(ker T)$= \frac{n}{2}$, dim(im T)$=$ dim $V-$ dim(ker T)$= n - \frac{n}{2} = \frac{n}{2}$, and for odd n, dim(ker T)$= \frac{n-1}{2}$, dim(im T)$= n - \frac{n-1}{2} = \frac{n+1}{2}$.

Problem 6: Let $T : V \to V$ be a linear operator on a vector space of dimension 2. Assume that T is not multiplication by a scalar. Prove that there is a vector v in V such that $(v, T(v))$ is a basis of V, and describe the matrix of T with respect to that basis.

Solution 6:

Since $T : V \to V$ is a linear operator on a vector space of dimension 2 and T is not multiplication by a scalar, therefore for a given vector $v \in V$, $T(v) \neq \lambda v$, where $\lambda \in F$, i.e., $\alpha v + \beta T(v) = 0$ only if $\alpha = 0$ and $\beta = 0$, i.e., $(v, T(v))$ are linearly independent.

Clearly, T and $T(v)$ are distinct because if not, then $T(v) = 1 \cdot v$, which is not possible. Also, since V has dimension 2, so it spans and hence $(v, T(v))$ is a basis for all $v \in V$.

Problem 7:

(a) Show that Hom (V, W) is a vector space under pointwise addition and scalar multiplication.

(b) Suppose $\mathbf{B} = \{v_1, \ldots, v_p\}$ is a basis for V and $\mathbf{C} = \{w_1, \ldots, w_q\}$ is a basis for W. For any linear map T from V to W, we constructed a q by p matrix with respect to given bases. Call this matrix $f(T)$. Show that f is an isomorphism from the vector space Hom (V, W) to the vector space of all q by p matrices with real entries.

(c) Take T_1 and T_2 in Hom (V, W) and S in Hom (W, U) for some vector space U. Show that (S composed with $(T_1 + T_2)$) = (S composed with T_1) + (S composed with T_2). State without proof analogous formula using R in Hom(U, V). In light of what we did in class (about composition of linear maps vis a vis their matrices) and part (b) above, these formulas translate into a property of matrix operations. State this property and explain very briefly.

Solution 7:

(a) Let $S, T \in$ Hom (V, W) be linear maps. So, for $v \in V$, we define

$$(S + T)(v) = S(v) + T(v)$$

So for $v_1, v_2, v \in V$, we have

$$\begin{aligned}(S + T)(v_1 + v_2) &= S(v_1 + v_2) + T(v_1 + v_2) \\ &= S(v_1) + S(v_2) + T(v_1) + T(v_2) \\ &= S(v_1) + T(v_1) + S(v_2) + T(v_2) \\ &= (S + T)(v_1) + (S + T)(v_2)\end{aligned}$$

and

$$(S + T)(\lambda v) = S(\lambda v) + T(\lambda v)$$

$$= \lambda S(v) + \lambda T(v)$$
$$= \lambda(S + T)(v)$$

Thus, $S + T$ is a linear map and $S + T \in \text{Hom } (V, W)$.
Now for $v \in V$, we define

$$(\lambda T)(v) = \lambda T(v)$$

So for $v_1, v_2, v \in V$, we have

$$(\lambda T)(v_1 + v_2) = \lambda(T(v_1 + v_2)$$
$$= \lambda\{T(v_1) + T(v_2)\}$$
$$= \lambda T(v_1) + \lambda T(v_2)$$
$$= (\lambda T)(v_1) + (\lambda T)(v_2)$$

and

$$(\lambda T)(\mu v) = \lambda T(\mu V)$$
$$= \lambda\mu T(v)$$
$$= \mu\lambda T(v)$$
$$= \mu(\lambda T)(v)$$

Thus, λT is a linear map and $\lambda T \in \text{Hom } (V, W)$.
The remaining axioms are clearly satisfied and hence Hom (V, W) is a vector space under pointwise addition and scalar multiplication.
(b) Given that $\mathbf{B} = \{v_1, \ldots, v_p\}$ is a basis for V and $\mathbf{C} = \{w_1, \ldots, w_q\}$ is a basis for W, so dim $\mathbf{B} = p$ and dim $\mathbf{C} = q$.
Now,

$$f(T) = \begin{bmatrix} \vdots & \vdots & & \vdots \\ T(v_1) & T(v_2) & \cdots & T(v_p) \\ \vdots & \vdots & & \vdots \end{bmatrix}$$

which is a $q \times p$ matrix.
If $f(T) = 0$, then $T(v_i) = 0 \; \forall i = 1, 2, \ldots, p$. But since v_i's are basis vectors, we have $T = 0$. So f is injective.
Now, let $R = (r_{jk})$ be any $q \times p$ matrix and we define

$$T(v_i) = \sum_{j=1}^{q} r_{jk} w_j$$

for $k = 1, 2, \ldots, n$. Then we have $f(T) = R$. So f is surjective.
Thus, f is bijective.
(c) We take $\dim V = p$, $\dim W = q$ and $\dim U = r$ with bases $(v_1, \ldots, v_p)$, $(w_1, \ldots, w_q)$ and $(u_1, \ldots, u_r)$ respectively.
Now we take $x = a_1 v_1 + \cdots + a_p v_p$.
So, $T_1(x) = a_1 T_1(v_1) + \cdots + a_p T_1(v_p)$ and $T_2(x) = a_1 T_2(v_1) + \cdots + a_p T_2(v_p)$.
Therefore,

$$T_1(x) + T_2(x) = a_1\{T_1(v_1) + T_2(v_1)\} + \cdots + a_p\{T_1(v_p) + T_2(v_p)\}$$

$$= a_1(T_1 + T_2)(v_1) + \cdots + a_p(T_1 + T_2)(v_p)$$

or,

$$(T_1 + T_2)(x) = T_1(x) + T_2(x)$$

Now,

$$S(T_1(x)) = a_1 S(T_1(v_1)) + \cdots + a_p S(T_1(v_p) \text{ and } S(T_2(x)) = a_1 S(T_2(v_1)) + \cdots + a_p S(T_2(v_p)$$

Therefore,

$$S(T_1(x)) + S(T_2(x)) = a_1(S(T_1 + T_2))(v_1) + \cdots + a_p(S(T_1 + T_2))(v_p)$$

or,

$$(S(T_1 + T_2))(x) = S(T_1(x)) + S(T_2(x))$$

or,

$$S \circ (T_1 + T_2) = S \circ T_1 + S \circ T_2$$

Problem 8:
(a) Let V, W, U be finite dimensional vector spaces, T a linear map from V to W and S linear map from W to U. Thus ST (i.e. the composition T followed by S) is a linear map from V to U. Prove using only abstract vector space language that rank(ST) is less than or equal to rank(S) as well as rank(T).
(b) Let A and B be matrices such that the product AB is defined. Prove using only matrices (without any mention of vector spaces) that rank(AB) is less than or equal to rank(A) as well as rank(B).
(c) Your proofs in (a) and (b) should be independent of each other. Now explain briefly but precisely how either of (a) and (b) can deduced from the other.
Solution:
(a) Given $T : V \to W$ and $S : W \to U$ are linear maps. So, $ST : V \to U$. Consider a matrix $M \in \mathbb{R}^{mn}$.
So, Rank(M)= dim(Range M), Rank(ST)= dim(Range ST) and Rank(S)=

dim (Range S) and Rank (T)= dim(Range T).
We have in general, if a vector space W is a subspace of a vector space V, then $\dim(W) \leq \dim(V)$.
Now for any vector $y \in$ Range ST, $\exists\, x$ such that $y = (ST)x = S(Tx) = Sz$, taking $z = Tx$. So, $y \in$ Range S, i.e., Range ST is a subset of Range S.
So, Rank (ST)= dim(Range ST)$\leq$ dim(Range S)=Rank (S). Also, $z \in$ Range T, so span(y) is a subspace of T.
Therefore, dim(span (y))= dim(U)$\leq$ dim(Range T) and hence dim(Range ST)$\leq$ Rank (T).

(b) Let $A = \begin{bmatrix} \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_q \\ \vdots & \vdots & & \vdots \end{bmatrix} \in \mathbb{R}^{pq}$, $B \begin{bmatrix} \vdots & \vdots & & \vdots \\ b_1 & b_2 & \cdots & b_r \\ \vdots & \vdots & & \vdots \end{bmatrix} \in \mathbb{R}^{qr}$.

So, $AB = \begin{bmatrix} \vdots & \vdots & & \vdots \\ b_1 A & b_2 A & \cdots & b_r A \\ \vdots & \vdots & & \vdots \end{bmatrix}$, i.e., the columns of AB are the linear combination of the columns of A, with B as scalar multiples. Therefore, column space of A is a subspace of the column space of A.
Therefore, Rank (AB)= dim(column space of AB)$\leq$ dim(column space of A)= Rank (A).

Also, $A = \begin{bmatrix} \cdots & a'_1 & \cdots \\ \cdots & a'_2 & \cdots \\ & \vdots & \\ \cdots & a'_p & \cdots \end{bmatrix}$ and $A = \begin{bmatrix} \cdots & b'_1 & \cdots \\ \cdots & b'_2 & \cdots \\ & \vdots & \\ \cdots & b'_q & \cdots \end{bmatrix}$.

So, $AB = \begin{bmatrix} \cdots & a'_1 B & \cdots \\ \cdots & a'_2 B & \cdots \\ & \vdots & \\ \cdots & a'_p B & \cdots \end{bmatrix}$, i.e., the rows of AB are linear combination of the rows B with A as scalar multiples. Therefore, the row space of AB is a subspace of the row space of B.
Therefore, Rank (AB)= dim(row space of AB)$\leq$ dim(row space of B)= Rank (B).