Algebra 1 HW 2

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Problem 1:
Which of the following subsets is a subspace of the vector space $F^{n\times n}$ of $n\times n$ matrices with coefficients in F?
(a) symmetric matrices ($A^t = A$),
(b) invertible matrices,
(c) upper triangular matrices.
Find a basis for the space of $n \times n$ symmetric matrices ($A^t = A$).
Problem 2:
Prove that the three functions x^2, $\cos x$, and e^x are linearly independent.
Problem 3:
Let $(X_1, \ldots, X_m)$ and $(Y_1, \ldots, Y_n)$ be bases for $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. Do the mn matrices $X_i Y_j^t$ form a basis for the vector space $\mathbb{R}^{m\times n}$ of all $m \times n$ matrices?
Problem 4:
(a) Prove that the set $\mathbf{B} = ((1, 2, 0)^t, (2, 1, 2)^t, (3, 1, 1)^t)$ is a basis of $\mathbb{R}^3$.
(b) Find the coordinate vector of the vector $v = (1, 2, 3)^t$ with respect to this basis.
Problem 5:
Let $\mathbf{B} = (v_1, \ldots, v_n)$ be a basis of a vector space V. Prove that one can get from $\mathbf{B}$ to any other basis $\mathbf{B}'$ by a finite sequence of steps of the following types:
(i) Replace v_i by $v_i + av_j$, $i \neq j$, for some a in F,
(ii) Replace v_i by cv_i for some $c \neq 0$,
(iii) Interchange v_i and v_j.
Problem 6:
Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 10 & 20 & 30 & 40 & 50 \\ 3 & 6 & 7 & 18 & 14 \end{bmatrix}$. Using only row operations on A, find a basis for **(i)** Null space of A, **(ii)** Column space of A, and **(iii)** Row space of A.
Problem 7:
Let A be an $r \times c$ matrix with the associated function $f(x) = Ax$. Show that any two the following three statements implies the third.
(i) A is a square matrix, i.e., $r = c$.
(ii) $Ax = 0$ has only the trivial solution (i.e., f is injective).
(iii) $Ax = b$ has a solution for every vector b (i.e., f is surjective).
Problem 8:
Let A be a square matrix. Show that the following are equivalent.
(a) A is invertible.
(b) The function $f(x) = Ax$ is a bijection.
(c) There exists matrix L such that $LA = I$.

(d) There exists matrix R such that $AR = I$.
(e) RREF(A) = identity matrix I.
(f) A is a product of elementary matrices.
Solution 1:
(a) Let F_1 be the set of all symmetric matrices i.e., $F_1 = \{A \mid A^t = A\}$. For any $n \times n$ symmetric matrices $A = (a_{ij})$ and $B = (b_{kl})$ such that $a_{ij} = a_{ji}$ and $b_{kl} = b_{lk}$ $\forall i, j, k, l \in \{1, 2, \ldots, n\}$, we have

$$A + B = (a_{ij}) + (b_{kl}) = (a_{ij} + b_{kl}) = (a_{ji} + b_{lk}) = (A + B)^t$$

and

$$\lambda A = \lambda(a_{ij}) = (\lambda a_{ij}) = (\lambda a_{ji}) = (\lambda A)^t$$

for any $\lambda \in F$. So, $A + B \in F_1$ and $\lambda A \in F_1$. So, F_1 is a subspace of $F^{n\times n}$.
(b) Let F_2 be the set of all invertible matrices. Then, the $n \times n$ square matrices $A = (a_{ij})$ and $-A = (-a_{ij})$, with not all of a_{ij} zeroes, are invertible. But then we have $A + (-A) = 0$, the zero matrix, which is not invertible. Thus, F_2 is not a subspace of $F^{n\times n}$
(c) Let F_3 be the set of all upper triangular matrices. Let $A, B \in F_3$ and

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ 0 & a_{22} & \ldots & a_{2n} \\ \vdots & & & \vdots \\ 0 & 0 & \ldots & a_{nn} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ 0 & b_{22} & \ldots & b_{2n} \\ \vdots & & & \vdots \\ 0 & 0 & \ldots & b_{nn} \end{bmatrix}$$

So,

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \ldots & a_{1n} + b_{1n} \\ 0 & a_{22} + b_{22} & \ldots & a_{2n} + b_{2n} \\ \vdots & & & \vdots \\ 0 & 0 & \ldots & a_{nn} + b_{nn} \end{bmatrix}$$

and

$$\lambda A = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \ldots & \lambda a_{1n} \\ 0 & \lambda a_{22} & \ldots & \lambda a_{2n} \\ \vdots & & & \vdots \\ 0 & 0 & \ldots & \lambda a_{nn} \end{bmatrix}$$

which are upper triangular matrices. So, $A + B \in F_3$ and $\lambda A \in F_3$. So, F_3 is a subspace of $F^{n\times n}$
Solution 2:
We assume, to the contrary, that x^2, $\cos x$ and e^x are linearly dependent i.e., $\exists\ \alpha, \beta, \gamma \in \mathbb{R}$, not all zeroes, such that $\alpha x^2 + \beta e^x + \gamma \cos x = 0$ $\forall x$. Let $f(x) = x^2 + \beta e^x + \gamma \cos x$. We have, $f(x) = 0 \Rightarrow$ all coefficients of $f(x)$

are zeroes, i.e., $\alpha = \beta = \gamma = 0$, which contradicts that x^2, $\cos x$ and e^x are linearly dependent. Thus, x^2, $\cos x$ and e^x are linearly independent.

Solution 3:

Since $(X_1, \dots, X_m)$ and $(Y_1, \dots, Y_n)$ are bases for $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, so they are linearly independent and they span. We first prove that the mn matrices $X_i Y_j^t$ are linearly independent. So, assuming wlog that $i < j$ and for any $i < k < j$, we take $c_i = 1$

$$\sum_j \sum_i c_i d_j X_i Y_j = \sum_i \sum_j c_i X_i d_j Y_j = \sum_i c_i X_i \sum_j d_j Y_j \neq 0$$

since X_i's and Y_j's form a basis and hence linearly independent. Thus, the mn matrices $X_i Y_j^t$ are also linearly independent.

Solution 4:

(a) Suppose for some x, y, z,

$$x \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + y \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + z \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e., we have a system of three equations

$$x + 2y + 3z = 0, 2x + y + z = 0, 2y + z = 0$$

To solve the system, we consider the augmented matrix

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right]$$

Using row operations $R_2 \to R_2 - 2R_1, R_1 \to R_1 - R_3$, we have the matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & -3 & -5 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right]$$

Using row operation $R_3 \to 3R_3 + 2R_2$, we have the matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & -3 & -5 & 0 \\ 0 & 0 & -7 & 0 \end{array}\right]$$

Using row operations $R_2 \to -\frac{1}{3}R_2, R_3 \to -\frac{1}{7}R_3$, we have the matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & \frac{5}{3} & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Now using row operations $R_1 \to R_1 - 2R_3, R_2 \to R_2 - \frac{5}{3}R_2$, we have the matrix (with A in RREF)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Thus, $x = y = z = 0$ and so $\mathbf{B}$ is linearly independent. Also, since RREF(A) has a pivot in every row, so it spans $\mathbb{R}^3$ and hence $\mathbf{B}$ is a basis of $\mathbb{R}^3$.

(b) Suppose for some x, y, z,

$$x \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + y \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + z \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

i.e., we have a system of three equations

$$x + 2y + 3z = 1, 2x + y + z = 2, 2y + z = 3$$

To solve the system, we consider the augmented matrix

$$[A|P] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 1 & 1 & 2 \\ 0 & 2 & 1 & 3 \end{array}\right]$$

Using row operations $R_2 \to R_2 - 2R_1, R_1 \to R_1 - R_3$, we have the matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -2 \\ 0 & -3 & -5 & 0 \\ 0 & 2 & 1 & 3 \end{array}\right]$$

Using row operation $R_3 \to 3R_3 + 2R_2$, we have the matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -2 \\ 0 & -3 & -5 & 0 \\ 0 & 0 & -7 & 9 \end{array}\right]$$

Using row operations $R_2 \to -\frac{1}{3}R_2, R_3 \to -\frac{1}{7}R_3$, we have the matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -2 \\ 0 & 1 & \frac{5}{3} & 0 \\ 0 & 0 & 1 & -\frac{9}{7} \end{array}\right]$$

Now using row operations $R_1 \to R_1 - 2R_3, R_2 \to R_2 - \frac{5}{3}R_3$, we have the matrix (with A in RREF)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{4}{7} \\ 0 & 1 & 0 & \frac{15}{7} \\ 0 & 0 & 1 & -\frac{9}{7} \end{array}\right]$$

So, the coordinate vector of v is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{4}{7} \\ \frac{15}{7} \\ -\frac{9}{7} \end{bmatrix} = \left(\frac{4}{7}, \frac{15}{7}, -\frac{9}{7}\right)^t$$

Solution 5:

(i) Given that $\mathbf{B} = (v_1, \ldots, v_n)$ is a basis of a vector space V. If we replace v_i by $v_i + av_j, i \neq j$ for some a in F, then for some $c_1, c_2, \ldots, c_n$ and assuming wlog that $i < j$, we have

$$c_1v_1 + c_2v_2 + \cdots + c_i(v_i + av_j) + \cdots + c_jv_j + \cdots c_nv_n$$

$$= c_1v_i + c_2v_2 + \cdots + c_iv_i + \cdots + (c_ia + c_j)v_j + \cdots + c_nv_n$$

and hence the span of $\mathbf{B}$ and $\mathbf{B}'$ are equal. So we can get from $\mathbf{B}$ to $\mathbf{B}'$ by the given step.

(ii) If we replace v_i by cv_i for some $c \neq 0$, then for some $c_1, c_2, \ldots, c_n$, we have

$$c_1v_1 + c_2v_2 + \cdots + c_i(cv_i) + \cdots + c_nv_n$$

$$= c_1v_1 + c_2v_2 + \cdots + (cc_i)v_i + \cdots + c_nv_n$$

and hence the span of $\mathbf{B}$ and $\mathbf{B}'$ are equal. So we can get from $\mathbf{B}$ to $\mathbf{B}'$ by the given step.

(iii) If we interchange v_i and v_j, assuming wlog that $i < j$, we have

$$c_1v_1 + c_2v_2 + \cdots + c_iv_j + \cdots + c_jv_i \cdots + c_nv_n$$

$$= c_1v_1 + c_2v_2 + \cdots + c_jv_i + \cdots + c_iv_j \cdots + c_nv_n$$

and hence the span of $\mathbf{B}$ and $\mathbf{B}'$ are equal. So we can get from $\mathbf{B}$ to $\mathbf{B}'$ by the given step.

Solution 6:

(i) Null space of $A = \{X \mid AX = 0\}$. We consider the augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 0 \\ 10 & 20 & 30 & 40 & 50 & 0 \\ 3 & 6 & 7 & 18 & 14 & 0 \end{array}\right]$$

Using row operations $R_2 \to R_2 - 10R_1, R_1 \to R_1 - 3R_3$, we have the matrix

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & -14 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 6 & -1 & 0 \end{array}\right]$$

Interchanging rows R_2 and R_3, we have the matrix (A reduced to RREF)

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & -14 & 8 & 0 \\ 0 & 0 & 1 & 6 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, the solution set is

$$\begin{bmatrix} -2x_2 + 14x_4 - 8x_5 \\ x_2 \\ -6x_4 + x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 14 \\ 0 \\ -6 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -8 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

So, a basis for the null space of A is $((-2,1,0,0,0)^t, (14,0,-6,1,0)^t, (-8,0,1,0,1)^t)$.
(ii) Since the 1^{st} and 3^{rd} columns of RREF(A) contain pivots, so the set of the 1^{st} and 3^{rd} columns of A i.e., $((1,10,3)^t, (3,30,7)^t)$ is a basis for the column space of A.
(iii) Since the 1^{st} and 2^{nd} rows of RREF(A) contain pivots, so a basis for the row space is $((1,2,0,-14,8), (0,0,1,6,-1))$
Solution 7:
Let A be an $r \times c$ matrix with the associated function $f(x) = Ax$.
We first prove that **(i)** and **(ii)** implies **(iii)**.
We know that $Ax = 0$ has only the trivial solution (i.e., f is injective) means that every column of RREF(A) has a pivot. Since A is a square matrix, i.e., $r = c$, it means that every row of RREF(A) also has a pivot. Thus, f is surjective.
Now we prove that **(i)** and **(iii)** implies **(ii)**.
We know that $Ax = b$ has a solution for every vector b (i.e., f is surjective) means that every row of RREF(A) has a pivot. Since A is a square matrix, i.e., $r = c$, it means that every column of RREF(A) also has a pivot. Thus, f is injective.
Now we prove that **(ii)** and **(iii)** implies **(i)**.
We know that $Ax = 0$ has only the trivial solution (i.e., f is injective) means that every column of RREF(A) has a pivot. Again, $Ax = b$ has a solution for every vector b (i.e., f is surjective) means that every row of RREF(A)

has a pivot. Thus, A is a square matrix i.e., $r = c$.

Solution 8:

Clearly, **(a)⇒(c)** and **(d)**.

We prove that **(a)⇒(b)**.

For any matrices A, B (such that AB exists), we define $f_A(x) = Ax$ and $f_B(x) = Bx$. Then $AB = I \Rightarrow f_A \circ f_B = f_I$, identity map from $\mathbb{R}^r \to \mathbb{R}^r$ and $BA = I \Rightarrow f_B \circ f_A = f_I$, identity map from $\mathbb{R}^c \to \mathbb{R}^c$. These equations say that f_A and f_B are both bijective.

Now we prove that **(a)⇒(e)**.

Since **(a)⇒(b)**, so we can use the fact that f is bijective, which means that every row and every column of RREF(A) has a pivot. Thus, RREF$(A) = I$, the identity matrix.

Now we prove that **(a)⇒(f)**.

Each row operation that used to reduce A to I can be represented by an elementary matrix. Suppose n row operations are required to reduce A to I. So,

$$E_n \cdots E_2 E_1 A = I$$

Since A is invertible and inverse of a matrix is unique if it exists, so

$$A^{-1} = E_n \cdots E_2 E_1$$

Since the inverse of elementary matrices are also elementary matrices, so we write A as a product of elementary matrices as

$$A = (A^{-1})^{-1} = (E_n \cdots E_2 E_1)^{-1} = {E_1}^{-1} {E_2}^{-1} \cdots {E_n}^{-1}$$

Clearly **(e)⇒ (c)**.

Now we prove **(b)⇒ (a)**.

Since f is bijective, so every row and every column of RREF(A) has a pivot, which implies that RREF$(A) = I$. Since **(e)⇒ (c)**, so $\exists$ a matrix B, which is the product of elementary matrices, such that $BA = I$. Multiplying to the left by B^{-1}, we have $A = B^{-1}$ i.e., A is invertible.

Since **(b)** implies **(a)** and **(a)** implies the others, so **(b)** implies the others. Clearly, **(c)** and **(d)** implies **(a)** and so they imply the others. Also, **(e)** implies **(c)**, so it implies the others. Also since **(f)** implies **(a)**, so it implies the others.