Algebra 1 HW 1

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Problem 1.1:

(i) Compute $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}^n$.

Let D be the diagonal matrix with diagonal entries $d_1, \dots, d_n$, and let $A = (a_{ij})$ be an arbitrary $n \times n$ matrix. Compute the products DA and AD.

(ii) A square matrix A is *nilpotent* if $A^k = 0$ for some $k > 0$. Prove that if A is nilpotent, then $I + A$ is invertible. Do this by finding the inverse.

(iii) Find infinitely many matrices B such that $BA = I_2$ when

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}$$

and prove that there is no matrix C such that $AC = I_3$.

(iv) Find each 2×2 matrix that commutes with all 2×2 matrices.

Problem 1.2:

(i) Find all solutions of the system of equations $AX = B$ when

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 3 & 0 & 0 & 4 \\ 1 & -4 & -2 & 2 \end{bmatrix} \text{ and } B = \textbf{(a)} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \textbf{(b)} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \textbf{(c)} \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

(ii) The matrix below is based on the Pascal triangle. Find its inverse.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix}$$

(iii) Prove that if a product AB of $n \times n$ matrices is invertible, so are the factors A and B.

Problem 1.3: The *trace* of a square matrix is the sum of its diagonal entries:

$$\operatorname{trace} A = a_{11} + a_{22} + \cdots + a_{nn}.$$

Show that $\operatorname{trace}(A + B) = \operatorname{trace} A + \operatorname{trace} B$, that $\operatorname{trace} AB = \operatorname{trace} BA$, and that if B is invertible, then $\operatorname{trace} A = \operatorname{trace} BAB^{-1}$.

Show that the equation $AB - BA = I$ has no solution in real $n \times n$ matrices A and B.

Solution 1.1

(i) By the method of matrix multiplication, we have

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1\times 1 + a\times 0 & 1\times b + a\times 1 \\ 0\times 1 + 1\times 0 & 0\times b + 1\times 1 \end{bmatrix} = \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix}$$

Claim: $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & na \\ 0 & 1 \end{bmatrix}$

Proof: We use induction.

Clearly for $n = 1$, $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}^1 = \begin{bmatrix} 1 & 1a \\ 0 & 1 \end{bmatrix}$ is true.

We assume that the statement is true for $n = k$, i.e.,

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}^k = \begin{bmatrix} 1 & ka \\ 0 & 1 \end{bmatrix}$$

Therefore,

$$\begin{aligned}\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}^{k+1} &= \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}^k \\ &= \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & ka \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1\times 1 + a\times 0 & 1\times ka + a\times 1 \\ 0\times 1 + 1\times 0 & 0\times ka + 1\times 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & (k+1)a \\ 0 & 1 \end{bmatrix}\end{aligned}$$

Thus, the statement is also true for $n = k + 1$. Hence, by the method of induction, we have $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & na \\ 0 & 1 \end{bmatrix}$.

Now, if D is the diagonal matrix with diagonal entries $d_1, \ldots, d_n$, and $A = (a_{ij})$ is an arbitrary $n \times n$ matrix, then the required products are $DA = (d_i a_{ij})$ and $AD = (a_{ij} d_j)$, which are again $n \times n$ matrices. □

(ii) If A is a nilpotent matrix, then $A^k = 0$ for some $k > 0$.

Let $B := I - A + A^2 - \cdots + (-1)^{k-1}A^{k-1}$.

Therefore,

$(I + A)B = (I + A)\left\{I - A + A^2 - \cdots + (-1)^{k-1}A^{k-1}\right\}$

$= \left\{I - A + A^2 - \cdots + (-1)^{k-1}A^{k-1}\right\} + \left\{A - A^2 + A^3 - \cdots + (-1)^{k-1}A^k\right\}$

$= I + (-1)^{k-1}A^k = I$

Similarly,

$B(I + A) = \left\{I - A + A^2 - \cdots + (-1)^{k-1}A^{k-1}\right\}(I + A)$

$= (I + A) - (A + A^2) + (A^2 + A^3) - \cdots + (-1)^{k-1}(A^{k-1} + A^k)$

$= I + (-1)^{k-1}A^k = I$
Therefore,

$$(I + A)B = B(I + A) = I$$

i.e., $I + A$ is invertible and it's inverse is $(I + A)^{-1} = B$. $\square$

(iii) Let $B = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$. Thus,

$$BA = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2a+b+c & 3a+2b+c \\ 2d+e+f & 3d+2e+f \end{bmatrix}$$

Now, if $BA = I_2$, i.e., if $\begin{bmatrix} 2a+b+c & 3a+2b+c \\ 2d+e+f & 3d+2e+f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then $b + c = 1 - 2a$ and $2b + c = -3a \Rightarrow b = -1 - a$ and $c = 2 - a$. Also, $e + f = -2d$ and $2e + f = 1 - 3d \Rightarrow e = 1 - d$ and $f = -1 - d$. Therefore, $\exists$ infinitely many matrices $B = \begin{bmatrix} a & -1-a & 2-a \\ d & 1-d & -1-d \end{bmatrix}$ such that $BA = I_2$.
Again, let $C = \begin{bmatrix} p & q & r \\ s & t & u \end{bmatrix}$. Thus,

$$AC = \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} p & q & r \\ s & t & u \end{bmatrix} = \begin{bmatrix} 2p+3s & 2q+3t & 2r+3u \\ p+2s & q+2t & r+2u \\ p+s & q+t & r+u \end{bmatrix}$$

Suppose $AC = I_3$, i.e., $\begin{bmatrix} 2p+3s & 2q+3t & 2r+3u \\ p+2s & q+2t & r+2u \\ p+s & q+t & r+u \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
In particular, $p + 2s = p + s = 0 \Rightarrow p = s = 0$. But this gives $2p+3s = 0$, a contradiction. Therefore, there is no matrix C such that $AC = I_3$. $\square$

(iv) Let $P = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ be a matrix that commutes with any arbitrary matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then we have

$$AP = PA$$

or,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

or,

$$\begin{bmatrix} ap+br & aq+bs \\ cp+cr & dq+ds \end{bmatrix} = \begin{bmatrix} ap+cq & bp+dq \\ ar+cs & br+ds \end{bmatrix}$$

or,

$$br = cq,\ aq+bs = bp+dq,\ cp+cr = ar+cs$$

But since the equation $br = cq$ is true for any arbitrary values of b and c, so $q = r = 0$.
Since $q = 0$, so $aq + bs = bp + dq \Rightarrow bs = bp \Rightarrow s = p$, if $b \neq 0$. But even if $b = 0$ and $s = p$, the equation $bs = bp$ is still true.
Since $r = 0$, so $cp + cr = ar + cs \Rightarrow cp = cs \Rightarrow p = s$, if $c \neq 0$. But even if $c = 0$ and $p = s$, the equation $bs = bp$ is still true.
Thus, $P = \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix} = p \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = pI_2$ is a matrix that commutes with all 2×2 matrices. □

Solution 1.2

(i) **(a)** We consider the augmented matrix

$$[A|B] = \left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 0 \\ 3 & 0 & 0 & 4 & 0 \\ 1 & -4 & -2 & 2 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - 3R_1} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 0 \\ 0 & -6 & -3 & 1 & 0 \\ 1 & -4 & -2 & 2 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_1} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 0 \\ 0 & -6 & -3 & 1 & 0 \\ 0 & -6 & -3 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 0 \\ 0 & -6 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \to -\frac{1}{6}R_2} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - 2R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{4}{3} & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, for arbitrary x_3, x_4, $x_1 = -\frac{4}{3}x_4$ and $x_2 = -\frac{1}{2}x_3 + \frac{1}{6}x_4$.

(b) We consider the augmented matrix

$$[A|B] = \left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 1 \\ 3 & 0 & 0 & 4 & 1 \\ 1 & -4 & -2 & 2 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - 3R_1} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 1 \\ 0 & -6 & -3 & 1 & -2 \\ 1 & -4 & -2 & 2 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_1} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 1 \\ 0 & -6 & -3 & 1 & -2 \\ 0 & -6 & -3 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 1 \\ 0 & -6 & -3 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

The last row implies that $0 = 1$. So the given system of equations $AX = B$ is inconsistent, i.e., has no solution.

(c) We consider the augmented matrix

$$[A|B] = \left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 0 \\ 3 & 0 & 0 & 4 & 2 \\ 1 & -4 & -2 & 2 & 2 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - 3R_1} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 0 \\ 0 & -6 & -3 & 1 & 2 \\ 1 & -4 & -2 & 2 & 2 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_1} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 0 \\ 0 & -6 & -3 & 1 & 2 \\ 0 & -6 & -3 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 0 \\ 0 & -6 & -3 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \to -\frac{1}{6}R_2} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{6} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - 2R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{4}{3} & \frac{2}{3} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{6} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, for arbitrary x_3, x_4, $x_1 = \frac{2}{3} - \frac{4}{3}x_4$ and $x_2 = -\frac{1}{3} - \frac{1}{2}x_3 + \frac{1}{6}x_4$.
□

(ii) Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix}$. Thus,

$$[A|I_5] = \left[\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Using row operations $R_5 \to R_5 - R_4$, $R_4 \to R_4 - R_3$, $R_3 \to R_3 - R_2$, $R_2 \to R_2 - R_1$, we get the matrix

$$\left[\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 3 & 3 & 1 & 0 & 0 & 0 & -1 & 1 \end{array}\right]$$

Using row operations $R_5 \to R_5 - R_4$, $R_4 \to R_4 - R_3$, $R_3 \to R_3 - R_2$, we get the matrix

$$\left[\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 1 & -2 & 1 \end{array}\right]$$

Using row operations $R_5 \to R_5 - R_4$, $R_4 \to R_4 - R_3$, we get the matrix

$$\left[\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 3 & -3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 & 3 & -3 & 1 \end{array}\right]$$

Using row operation $R_5 \to R_5 - R_4$, we get the matrix

$$\left[\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 3 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -4 & 6 & -4 & 1 \end{array}\right] = [I_5|A^{-1}]$$

Therefore, the inverse of the matrix A is

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ -1 & 3 & -3 & 1 & 0 \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix}.$$

□

(iii) We shall first prove the following Lemma.

Lemma: If P and Q are square matrices such that $PQ = I$, then $QP = I$.
Proof: Let $x_1, x_2, \dots, x_n$ be a basis of a space. First we prove that $Qx_1, Qx_2, \dots, Qx_n$ is also a basis by proving that they are linearly independent. We assume, to the contrary, that they are not, i.e., $\exists\ c_1, c_2, \dots c_n$, not all zero, such that

$$c_1 Qx_1 + c_2 Qx_2 + \cdots + c_n Qx_n = 0$$

Multiplying this equation from the left by P, we have

$$c_1 PQx_1 + c_2 PQx_2 + \cdots c_n PQx_n = 0$$

Using $PQ = I$, we have

$$c_1 x_1 + c_2 x_2 + \cdots c_n x_n = 0$$

i.e., the x_i's are linearly dependent, which is a contradiction to the fact that x_i's form a basis. Thus, Qx_i's form a basis.
Since $Qx_1, Qx_2, \dots, Qx_n$ is a basis, every vector y can be represented as a linear combination of those vectors, i.e., for any vector y there exists some vector x such that $Qx = y$.
Now, proving $QP = I$ is same as proving that for any vector y, we have $QPy = y$. Since we have proved that for any vector y there exists some vector x such that $Qx = y$. Thus, for any vector y, we have

$$QPy = QPQx = QIx = Qx = y$$

□

Since the $n \times n$ matrix AB is invertible, so $\exists$ a $n \times n$ matrix C such that $(AB)C = I$. By associativity of product of matrices, we have $A(BC) = I$. By the above Lemma, we have $(BC)A = I$. Therefore, A is invertible and $A^{-1} = BC$, a $n \times n$ matrix.
Also, we have $C(AB) = I$. By associativity of product of matrices, we have $(CA)B = I$. Again, by the above Lemma, we have $B(CA) = I$. Therefore, B is also invertible and $B^{-1} = CA$, a $n \times n$ matrix. $\square$

Solution 1.3: Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & & & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{bmatrix}$$

Therefore,

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \vdots & & & \vdots \\ a_{n1} + b_{n1} & a_{n2} + b_{n2} & \dots & a_{nn} + b_{nn} \end{bmatrix}$$

So, $\text{trace}\,(A+B) = (a_{11} + b_{11}) + (a_{12} + b_{12}) + \cdots (a_{nn} + b_{nn})$
$= (a_{11} + a_{12} + \cdots + a_{nn}) + (b_{11} + b_{12} + \cdots + b_{nn})$
$= \text{trace}\, A + \text{trace}\, B$

Now,

$$AB = \begin{bmatrix} \sum_{i=1}^{n} a_{1i}b_{i1} & \sum_{i=1}^{n} a_{1i}b_{i2} & \dots & \sum_{i=1}^{n} a_{1i}b_{in} \\ \sum_{i=1}^{n} a_{21}b_{i1} & \sum_{i=1}^{n} a_{2i}b_{i2} & \dots & \sum_{i=1}^{n} a_{2i}b_{in} \\ \vdots & & & \vdots \\ \sum_{i=1}^{n} a_{ni}b_{i1} & \sum_{i=1}^{n} a_{ni}b_{i2} & \dots & \sum_{i=1}^{n} a_{ni}b_{in} \end{bmatrix}$$

and

$$BA = \begin{bmatrix} \sum_{i=1}^{n} a_{i1}b_{1i} & \sum_{i=1}^{n} a_{i2}b_{1i} & \dots & \sum_{i=1}^{n} a_{in}b_{1i} \\ \sum_{i=1}^{n} a_{i1}b_{21} & \sum_{i=1}^{n} a_{i2}b_{2i} & \dots & \sum_{i=1}^{n} a_{in}b_{2i} \\ \vdots & & & \vdots \\ \sum_{i=1}^{n} a_{i1}b_{ni} & \sum_{i=1}^{n} a_{i2}b_{ni} & \dots & \sum_{i=1}^{n} a_{in}b_{ni} \end{bmatrix}$$

$$
\begin{aligned}
\text{So, trace } AB &= \sum_{i=1}^{n} a_{1i}b_{i1} + \sum_{i=1}^{n} a_{2i}b_{i2} + \cdots + \sum_{i=1}^{n} a_{ni}b_{in} \\
&= \sum_{k=1}^{n}\sum_{i=1}^{n} a_{ki}b_{ik} \\
&= \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ki}b_{ik} \\
&= \sum_{k=1}^{n}\sum_{i=1}^{n} a_{ik}b_{ki} \\
&= \sum_{i=1}^{n} a_{i1}b_{1i} + \sum_{i=1}^{n} a_{i2}b_{2i} + \cdots + \sum_{i=1}^{n} a_{in}b_{ni} \\
&= \text{trace } BA
\end{aligned}
$$

Now,

$$\text{trace } BAB^{-1} = \text{trace } (BA)B^{-1} = \text{trace } B^{-1}(BA) = \text{trace } (B^{-1}B)A = \text{trace } IA = \text{trace } A$$

To prove the second part of the problem, we first prove that

$$\text{trace } (A - B) = \text{trace } A - \text{trace } B$$

We have,

$$\text{trace } (A - B) = \text{trace} = \text{trace } (A + (-B)) = \text{trace } A + \text{trace } (-B)$$

It is clear that trace $(-B) = -$trace B, so trace $(A - B) =$ trace $A -$ trace B.
Now, if the equation $AB - BA = I$ has a solution in real $n \times n$ matrices A and B, then

$$\text{trace } (AB - BA) = \text{trace } I$$

Now using the fact that trace $(A - B) =$ trace $A -$ trace B and trace $I = n$, we have

$$\text{trace } AB - \text{trace } BA = n$$

Since trace $AB =$ trace BA, so we have $n = 0$, a contradiction.
Hence, the equation $AB - BA = I$ has no solution in real $n \times n$ matrices A and B. □