Algebra 1 Quiz 2 Your roll number: ______________

Notation: Given a field F, as usual $e_1, e_2, \ldots, e_n$ denotes the standard basis of F^n. You may use this in your answers too. Unless specified otherwise, the underlying field is $\mathbb{R}$.

$J(\lambda, k)$ denotes the $k \times k$ matrix in which each of the k diagonal entries is λ, each of the $k-1$ entries in slots $(i, i+1)$ is 1 and all other entries are 0. As a shorthand notation, one can write a general matrix in Jordan form in a single line as direct sum of various $J(\lambda, k)$, i.e.,

$$\Big(J(\lambda_1, k_1) \oplus J(\lambda_1, k_2) \oplus \cdots\Big) \oplus \Big(\text{Jordan blocks for } \lambda_2\Big) \oplus \cdots$$

Any matrix obtained by reordering the basis is NOT considered a different Jordan form for the operator.

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1. Consider the operator T on $\mathbb{R}^2$ whose matrix with respect to the basis v_1, v_2 given below is the matrix A given below. Write a matrix M such that $T([a,b]^t) = M\ [a,b]^t$ = product of matrix M with the column vector $[a,b]^t$. You are allowed and encouraged to write your answer as a product of matrices as long as each of the constituent matrices is written explicitly, i.e., all entries of each matrix should be clearly written out. There should no need to do any further calculations to figure out entries of the matrices you write.

$v_1 = \begin{bmatrix} 4 \\ 11 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, A = \begin{bmatrix} 19 & 11 \\ 20 & 22 \end{bmatrix}$. Then $T(\begin{bmatrix} a \\ b \end{bmatrix}) = M \begin{bmatrix} a \\ b \end{bmatrix}$ for matrix $M =$

2. Find the characteristic polynomial of the linear operator defined by $e_1 \xrightarrow{T} e_2 \xrightarrow{T} e_3 \xrightarrow{T} ce_1 + be_2 + ae_3$.

Matrix of $T =$ characteristic polynomial of $T =$

3. Consider the operator T given by $e_1 \xrightarrow{T} e_2 \xrightarrow{T} e_3 \xrightarrow{T} e_1$. List all eigenvalues of T along with the multiplicity of each eigenvalue (= dimension of corresponding eigenspace) for the indicated field F. E.g., for the diagonal matrix with diagonal entries p, p, q, q, q your answer would be "eigenvalues with multiplicity are $p(2)$ and $q(3)$." For each F state if T is diagonalizable by circling the correct option and give a succinct reason.

If $F = \mathbb{C}$, eigenvalues with multiplicity for T are ________________________. T is / is NOT diagonalizable because

If $F = \mathbb{R}$, eigenvalues with multiplicity for T are ________________________. T is / is NOT diagonalizable because

If $F = \mathbb{F}_3$, eigenvalues with multiplicity for T are ________________________. T is / is NOT diagonalizable because

4. Diagonalize the matrix A given below, i.e. list vectors u_1 and u_2 forming a basis such that with respect this basis the matrix of the operator $v \to Av$ is the diagonal matrix D, which you should specify

$A = \begin{bmatrix} 5 & -3 \\ -6 & 2 \end{bmatrix}$ Answers: $u_1 =$ $u_2 =$ $D =$

5. Write the minimal polynomial of the matrix $\begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ and justify briefly.

6. For the 10×10 matrix $M = J(a,3) \oplus J(a,1) \oplus J(b,4) \oplus J(b,2)$ (where $a \neq b$), find the following

The characteristic polynomial =

The minimal polynomial =

A basis of the b-eigenspace

A basis of the generalized a-eigenspace

A basis of the kernel of $(M - b\,\mathrm{Id})^3$

7. Find all possible Jordan forms of a nilpotent 10×10 matrix N that satisfies the following: dimensions of kernels of the matrices N, N^2, N^3 are respectively 5,9,10. Write your answer(s) using the notation given at the beginning. Can any of the three numbers 5,9,10 be deduced from the other two? Explain briefly.

Extra. (a) Write a matrix with entries in $\mathbb{F}_2$ that does not have a Jordan form. How many such matrices of smallest possible size are there? Note that the quadratic polynomials with coefficients in $\mathbb{F}_2$ are x^2, x^2+1, x^2+x and x^2+x+1. (b) Show me if you completed the proof of Cayley's formula n^{n-2} for counting labeled trees on n vertices.