Algebra 1 Midterm

Explain everything. You may not use the word "determinant" unless the problem mentions it. Other than that (unless specified otherwise), you may use any standard result, but refer to it precisely and justify why it is applicable. All vector spaces are over $\mathbb{R}$.

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1. Do the following for the 4×4 matrix A whose columns are unknown vectors w_1, w_2, w_3, w_4 but which row reduces to the matrix written on the board. Find the following. If the given information is not enough to answer the question, state so. Give concise justification for everything.

   (a) A basis for the null space of A. (b) A basis for the row space of A. (c) w_4 as a linear combination of other w_i. (d) determinant of A. (e) Explicit equations in variables x_1, x_2, x_3, x_4 such that the column space of A is precisely the solution set of these equations. Give the smallest possible number of equations.

   (f) Now suppose W is a subspace of $\mathbb{R}^n$. Sketch how you can realize it as the set of solutions of a system of linear equations. What is the minimum number of equations needed? You may refer to any logic you used in part (e) without repeating it.

2. Let M be an $r \times c$ matrix. (a) Prove from first principles that columns of M span $\mathbb{R}^r$ if and only if rows of M are linearly independent. You may use ONLY definitions of span/linear independence along with standard facts about the use of row reduction. (b) True or false? "Columns of M are linearly independent if and only if rows of M span $\mathbb{R}^c$."

3. For vector spaces U, V, W, suppose $\dim(U) = 26$, $\dim(V) = 9$, $\dim(W) = 2022$. Let $U \xrightarrow{T} V \xrightarrow{S} W$ be linear maps. Answer (a) and then list all possible values of the dimension of each of the vector spaces in (b) through (d).

   (a) Write all containments that MUST be true involving the given sets. Write "None" if there are none. *You may write just the answer without any justification.* Recall that $\ker(T)$ stands for the kernel/null space of T and $\operatorname{im}(T)$ stands for the image of T.

   - $\ker(T)$ and $\ker(ST)$ (both subspaces of U).
   - $\ker(S)$ and $\operatorname{im}(T)$ (both subspaces of V).
   - $\operatorname{im}(S)$ and $\operatorname{im}(ST)$ (both subspaces of W).

   ST : U → W

   2022 − 26 = 2016

   (b) $\ker(T)$ 17,26 (c) $\ker(ST)$ [illegible],26 (d) $\operatorname{im}(ST)$ 0,26

4. Define the linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$ by the formula $T[a,b,c]^t = [5a + 7b + 9c, 11a + 13b + 15c]^t$. Write the matrix for T with respect to the standard bases for $\mathbb{R}^3$ and $\mathbb{R}^2$. Then find a basis for $\mathbb{R}^3$ and a basis for $\mathbb{R}^2$ with respect to which the matrix of T takes the standard form, i.e., in which there is an identity block on the top left corner and all other entries are 0.

5. Let $V =$ the set of $n \times n$ matrices. Consider the operator $S : V \to V$ defined by $S(A) = A + A^t$.

   (a) Give a basis for $K =$ the kernel of S. What is the dimension of K ?

   (b) Give a basis for $W =$ the image of S. What is the dimension of W ?

   (c) Show that $V = K \oplus W$. You should not need more than a few short sentences. Here $\oplus$ is the internal direct sum as defined by Artin. Alternatively, you can show that V is isomorphic to the external direct sum of K and W defined in class.

6. Short computational questions

(a) Verify that the change of basis matrix *from* the old basis $[7,5]^t, [4,3]^t$ of $\mathbb{R}^2$ *to* the standard basis is the matrix with *rows* $[3,-4]$ and $[-5,7]$. (Note that you do not have to find the change of basis matrix, you just need to verify it.) Find the coordinates of $[1,1]^t$ with respect to the basis $[7,5]^t, [4,3]^t$ of $\mathbb{R}^2$.

(b) Find the matrix of the following operator with respect the standard basis of $\mathbb{R}^2$: counterclockwise rotation by 90 degrees followed by reflection in the line $y = x$. Can you describe the overall map geometrically?

(c) A square matrix of unknown size is a product of elementary matrices that correspond to the following row operations.

- 26 scalings of individual rows by factors 2,3,. . . ,27
- 9 row interchanges
- 2022 operations of the third kind (add a multiple of one row to another)

The order of multiplication of these 2057 elementary matrices is unknown. If possible, find the determinant of A.