Algebra 1 Quiz 3 instructions

- As announced earlier, this (optional) quiz consists of critiquing your own in-class final as per the instructions below. Your final exam answers are available to you on moodle. I have attached the final exam again with minor comments and known typos corrected. Email me if you find any further issues.
- Solve the final again to your satisfaction and then write a critique of each of the solutions you uploaded earlier. Imagine that you are teaching someone else using your own final exam as an illustrative document. Write down, with the benefit of hindsight, what you think is the most edifying commentary – about mathematical content as well as writing. If that consists of "No changes needed" then that is fine. I will interpret that as you saying "This is a complete, correct and coherently written solution that can be published as a model solution for someone to learn from". But be sure your written answer meets that bar!
- Steps: (i) Annotate the already uploaded pdf file of each question with comments. Ideally you should do this on the same pdf file if you can. Otherwise write your commentary and include your earlier scan *below* that. (ii) Next, include any more material you would like me to see (e.g., corrected answer/new solution/more commentary), which you can either TeX (preferred) or write neatly and scan. (iii) At the very end, assign a score to your original solution to each problem as per the breakup given below.

Q1	5 + 3
Q2	6
Q3	6 + 4
Q4	3.5 + 2.5 + 2
Q5	10 (you decide the breakup)
Q6	3+3+4+2+2+4

  What I really want to see is your post-mortem of your own understanding at the time you wrote the final. Such self-reflection is HARD to do, but also worthwhile. Especially if your performance was not to your satisfaction, this is another opportunity to gain and demonstrate understanding without exam pressure. I will decide your score on quiz 3 based on the quality of linear algebra thinking that is reflected in your writing, so make it as incisive and coruscating as you can. It can be short and pithy.
- Submit your quiz 3 on moodle by uploading each question separately. Do NOT email it to me. The deadline is Sunday December 11, but in the spirit of an exam, I would like you to finish quiz 3 within 24 hours after starting it.

Algebra 1 final exam instructions

- **Put away your phone** and everything except your writing material.
- **New problem on new physical sheet of paper (not just new side)**. Write your roll number and problem number at the top of each sheet and staple all sheets corresponding to a *single* problem together. You will submit 6 such different sets at the end of the exam.
- **Submitting the exam**: (i) At the end of the exam (1 pm or earlier), take your phone, go back to your seat and <u>upload your exam on moodle within 30 minutes</u>. You should upload the answer to each problem as a separate file. It is your responsibility to ensure that everything is uploaded correctly. (ii) Once this is done (within 30 minutes), you must

turn in the physical answer papers as well in six different parts. Do NOT leave until getting verbal acknowledgment from me of your submissions.

- **Justify everything**. In all cases when you want to appeal to a theorem, the exact result being used must be clearly specified by name (e.g., dimension theorem) or by sufficient explanation. *I really REALLY care about the quality of your explanation.* You should too.

  You may appeal to any standard result proved in class with the following exceptions. You may not use a result if doing so renders the problem trivial. You may not mention *generalized* eigenspaces and facts about them unless the question mentions *generalized* eigenspaces and/or the Jordan form. You may not mention the Jordan form unless the question mentions the Jordan form. You may freely use definitions of eigenvalues/vectors/spaces, characteristic and minimal polynomials of an operator and diagonalizability. You may also use basic facts about these concepts that can be proved without using either of *generalized* eigenspace decomposition and the Jordan form. **Ask me in case of any doubt about these matters.**

- **Conventions and notation**: (i) Unless specified otherwise, the underlying field is $\mathbb{R}$. (ii) Given a field F, as usual $e_1, e_2, \ldots, e_n$ denotes the standard basis of F^n. You may use this in your answers too. (iii) $J(\lambda, k)$ denotes the $k \times k$ matrix in which each of the k diagonal entries is λ, each of the $k-1$ entries in slots $(i, i+1)$ is 1 and all other entries are 0. As a shorthand notation, one can write a general matrix in Jordan form in a single line as direct sum of various $J(\lambda, k)$, i.e.,

  $$\Big(J(\lambda_1, k_1) \oplus J(\lambda_1, k_2) \oplus \cdots\Big) \oplus \Big(\text{Jordan blocks for } \lambda_2\Big) \oplus \cdots$$

  Any matrix obtained by reordering the basis is NOT considered a different Jordan form for the operator.

Algebra 1 final exam

1. (a) Find with proof all real numbers c for which the matrix $A = \begin{bmatrix} 0 & c \\ 1 & 1 \end{bmatrix}$ is diagonalizable. For $c = 2$, diagonalize the matrix, i.e., find an explicit basis v_1, v_2 of $\mathbb{R}^2$ consisting of two column vectors with respect to which the matrix of the operator $v \mapsto Av$ is a diagonal matrix. (b) Consider the sequence f_i defined by $f_0 = 20, f_1 = 22, f_{i+2} = 2f_{i+1} + f_i$ for $i > 1$. Find a simple closed formula for f_n. If you have a doubt as to whether your formula is acceptable, ask me. (It may be simpler to prove that the formula has a specific form and then just solve for the parameters.) *Added after the exam: I meant to define $f_{i+2} = f_{i+1} + 2f_i$ for $i \geq 0$, thereby reducing the situation to part (a). Sorry for making it messy for you!*

2. Suppose a basis consisting of k column vectors $w_1, \ldots, w_k$ of a subspace W of F^n is given. Using the outline below or otherwise, crisply prove that W is the set of solutions of a system of linear equations in n variables, where the variables are written as a column vector $[x_1, \ldots x_n]^t$ in F^n.

Outline: Complete the given basis of W to get an ordered basis $B = (w_1, \ldots, w_k, u_{k+1}, \ldots, u_n)$ of F^n. Fix this basis for F^n and fix the *standard* basis of $F^?$ (the value of ? to be decided by you) so that the following are true:

(1) W is the kernel of a linear map $F^n \xrightarrow{\phi} F^?$ where ? has the least possible value. (2) The matrix of ϕ with respect to the two chosen bases of F^n and $F^?$ is $M =$ __________. Write the simplest possible matrix M in the sense that M should have the maximum possible number of zeros and should be valid over any field F.

Now write a system of equations $Ax = b$ whose solution set is precisely the subset W of F^n. Your answer must make it clear how to obtain all entries of the matrix A and the vector b using the given column vectors $w_1, \dots, w_k$. (The fact that any subspace W of a vector space V is the kernel of some linear map is "obvious" by taking the quotient map $V \to V/W$. The point of this question is to get explicit equations.)

3. (a) Let $p(x)$ be the characteristic polynomial of a linear operator $F^n \xrightarrow{T} F^n$, where $F = \mathbb{C}$ or $F = \mathbb{R}$. Using the following method or otherwise, prove that $p(T) =$ the 0 operator on F^n.

First consider the case where the matrix of T with respect to the standard basis is $A = (a_{ij})$ with diagonal entries $a_{ii} = \lambda_i$ (the entries λ_i may not all be distinct) and *all entries of A below the diagonal are* 0, i.e., $a_{ij} = 0$ for $i > j$. Complete the following sentences (write full sentences in your exam) and continue the reasoning to prove the result.

- $(A - \lambda_1 I)e_1 =$ __________.

  $(A - \lambda_2 I)e_2 =$ __________ and hence $(A - \lambda_1 I)(A - \lambda_2 I)e_2 =$ __________.

  Similarly __________$e_3 =$ __________ and hence __________.

  Continuing in this fashion we get that for each standard basis vector e_i, __________.

  Complete the proof for the T with the given matrix is A.

- Then, for $F = \mathbb{C}$, prove for arbitrary T that $p(T) =$ the 0 operator.

  Now let $F = \mathbb{R}$. Justify why you can now say that $p(T) = 0$.

(b) Let v_1, v_2, v_3, v_4 be a basis of V over some field F. Consider the linear operator S on V defined by

$$v_1 \xrightarrow{S} v_2 \xrightarrow{S} v_3 \xrightarrow{S} v_4 \xrightarrow{S} av_1 + bv_2 + cv_3 + dv_4,$$

where a, b, c, d are some (fixed) scalars from F. Find the minimal polynomial $m(x)$ of S from first principles.

4. (a) In the Euclidean space $\mathbb{R}^3$, find the unit vector u_1 in the direction of $v_1 = [1, 2, 2]^t$. Find a vector v_2 of form $[a, b, 0]^t$ where $a > 0$ and b are coprime integers and v_2 is orthogonal to v_1. Write an orthogonal matrix M whose first column is u_1.

(b) Write the matrix R of rotation by 30° around X-axis. Describe geometrically the operators $T(v) = MRM^{-1}v$ and $S(v) = M^{-1}RMv$. *Note added later: M in this part is the same M from part a.*

(c) Write a formula for the reflection S along v_1, i.e., reflection in the plane perpendicular to v_1. Your formula should be completely explicit so that for any given $v = [p, q, r]^t$, one can directly calculate $S(v)$ by plugging v into your formula. *Note added later: the reflection S in this part is not the same as S in part b. (It cannot be!) Sorry about the mixup.*

5. Consider the 25×25 matrix B in the Jordan form given as follows.

$$B = \Big(J(\pi, 4) \oplus J(\pi, 1)\Big) \oplus \Big(J(\sqrt{2}, 11) \oplus J(\sqrt{2}, 3) \oplus J(\sqrt{2}, 2)\Big) \oplus \text{unknown}.$$

It is also known that the determinant of B is 0 and B has exactly three eigenvalues.

(a) The span of $\{e_1, e_2, e_3, e_4\}$ is a B-invariant subspace of $\mathbb{R}^{25}$. List with proof all B-invariant subspaces of this subspace. Name your subspaces $W_1, W_2, \dots$ on separate lines and next to each W_i write a basis for it.

(b) How many B-invariant subspaces does the span of $\{e_6, \dots, e_{21}\}$ have (including 0 and the whole space)? *Note added later: (e) below gives away the answer!*

(c) Find all possibilities for the following: (i) dimension of the null space of B, (ii) dimension of the null space of B^2, (iii) the characteristic polynomial of B, (iv) the minimal polynomial of B, (v) all possibilities for the "unknown" part. *Note added later: the given order of subparts may not be logically the simplest.*

(e) This part is about arbitrary matrices in Jordan form. Find all $n \times n$ matrices M in Jordan form such that $\mathbb{R}^n$ has *no* proper nonzero M-invariant subspaces. Show that $\mathbb{R}^n$ has infinitely many M-invariant subspaces if M has an eigenspace of dimension greater than 1. Can you characterize Jordan forms for which $\mathbb{R}^n$ has only finitely many M-invariant subspaces? *Note added later: there was no part (d).*

6. Short problems with brief explanations. Do as many as you can.

(a) Suppose A is a $p \times q$ matrix with rows $R_1, \dots, R_p$ and columns $C_1, \dots, C_q$. Then $A\,[x_1, \dots, x_q]^t$ is a linear combination of the vectors __________ with weights respectively __________. For another matrix B of suitable size, the rows of AB are linear combinations of rows of __________ and columns of AB are linear combinations of columns of __________. Use this to show that the rank of AB is at most equal to the rank of A as well as the rank of B. Clearly state the basic facts about dimension that you are using.

(b) Show that for linear maps $V \xrightarrow{T} W \xrightarrow{S} U$, between finite dimensional vector spaces, dim ker $ST \leq$ dim ker $T+$ dim ker S. Give some sufficient conditions for equality. Can you give a good necessary and sufficient condition?

(c) The determinant is the unique map from $F^n \times \cdots \times F^n$ (taken n times) to F such that __________. State the characterization clearly. Using this succinctly show the following. (i) How one can deduce the effect of elementary row operations on the determinant (ii) $det(AB) = det(A)det(B)$ (iii) The usual formula for the 3×3 determinant. Do NOT write pages and pages, just crisply recapitulate the theory.

(d) On the vector space of polynomials in one variable x with real coefficients, define two operators by $S(p(x)) = xp(x)$ and $D(p(x)) = p'(x)$. Calculate $SD - DS$. The answer may seem to contradict a homework exercise you did, namely that traces of matrices AB and BA are equal. Resolve the apparent contradiction.

(e) If M and N are similar matrices, then they have the same characteristic polynomial and same minimal polynomial. Converse?

(f) Prove that the characteristic polynomial of T and the minimal polynomial of T have the same roots.

The rest is for your thinking only. Do not write any answers, but if you find elementary ways to do any of the following, tell me in January. (i) Characteristic polynomial of T factors completely if and only if the minimal polynomial factors completely. What is the most elementary proof you can find of this fact? (ii) For $n \times n$ matrices A and B, if A or B is invertible, then it is easy to see that AB and BA are similar and hence have the same characteristic polynomial. In fact AB and BA have the same characteristic polynomial even when neither is invertible. There is a trick proof, but alternatively, try to use the field of rational functions (look up the easy definition) to reduce to the easy case. (iii) Any square matrix over $\mathbb{C}$ is similar to its transpose. In fact the result is true over any field and the change of basis matrix can be chosen to be symmetric.

Further exploration: (iv) Cayley-Hamilton theorem is true over an arbitrary field. Look up a proof. (v) Can one define the determinant of an operator without using a matrix? Look up the exterior algebra. (vi) Note that the determinant of 2×2 skew symmetric matrix ($A^t = -A$) is the square of either off-diagonal entry. The determinant of any $2n \times 2n$ skew symmetric matrix is a square. Look up the Pfaffian. Also find the determinant of a skew symmetric matrix of *odd* size. A way to understand the answer is that in the odd case the associated bilinear form is degenerate. For this first understand the matrix of a bilinear form, how this matrix changes under a change of basis and how the matrix determines nondegeneracy of the form. Then look up the easy classification of skew symmetric forms over any field. (Classification of symmetric forms is harder depending on the field.)